Set 8 — Solutions —

(1) For $n \geq 0$, $(z-z_0)^n$ is analytic (no poles)

$\Rightarrow \oint (z-z_0)^n dz = 0$ around closed contour

For $n<0$, define $m=-n \Rightarrow (z-z_0)^n = (z-z_0)^{-m}$, $m>0$

Cauchy integral thm $\rightarrow \dfrac{(m-1)!}{2\pi i}\oint \dfrac{f(z)}{(z-z_0)^m}dz = \dfrac{d^{(m-1)}}{dz^{(m-1)}} f(z)\Big|_{z=z_0}$

Choose $f(z)=1 \Rightarrow d^{(m-1)}f/dz^{(m-1)} = 1$ for $m=1$, $=0$ for $m>1$

$$\Rightarrow \boxed{\oint (z-z_0)^n dz = 0,\ n \neq -1} \qquad \boxed{\oint (z-z_0)^{-1} dz = 2\pi i}$$

Text 15.6] (2) Let $z = e^{i\theta} \Rightarrow z^n = e^{in\theta} = \cos(n\theta) + i\sin(n\theta)$

Also, $\cos\theta = \frac{1}{2}(e^{i\theta}+e^{-i\theta}) = \frac{1}{2}(z+z^{-1})$

$$\Rightarrow \frac{\cos(n\theta)}{1-2a\cos\theta+a^2} = \mathrm{Re}\left[\frac{z^n}{1-a(z+z^{-1})+a^2}\right] = \mathrm{Re}\left[\frac{z^{n+1}}{(1+a^2)z-az^2-1}\right]$$

Now $dz = ie^{i\theta}d\theta = iz\,d\theta \Rightarrow d\theta = -i\,dz/z$

$$\Rightarrow \int_0^\pi \frac{\cos(n\theta)d\theta}{1-2a\cos\theta+a^2} = \frac{1}{2}\int_0^{2\pi}\frac{\cos(n\theta)d\theta}{1-2a\cos\theta+a^2} = \frac{1}{2}\mathrm{Re}\left[\oint \frac{+iz^n dz}{-(1+a^2)z+az^2+1}\right]$$

Poles occur where $-(1+a^2)z+az^2+1 = 0$

$$\Rightarrow z = \frac{1}{2a}\left[(1+a^2) \pm \sqrt{(1+a^2)^2-4a}\right] = \frac{1}{2a}\left[(1+a^2) \pm (1-a^2)\right]$$

$\Rightarrow z = a$ or $z = 1/a \rightarrow z=a$ outside unit circle for $a>1$

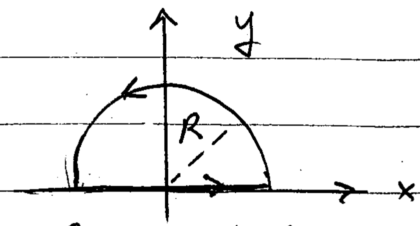
At $z=1/a$, residue $= \frac{i}{2}(1/a)^n(z-1/a)/[az^2-(1+a^2)z+1]\Big|_{z=1/a}$

Now $az^2-(1+a^2)z+1 = a(z-a)(z-1/a)$

$\Rightarrow 2\pi i \times \text{residue} = -\pi a^{-n}/[a(1/a-a)] = -\pi/a^n\,[1/(1-a^2)]$

$$\Rightarrow \boxed{\int_0^\pi \frac{\cos(n\theta)d\theta}{1-2a\cos\theta+a^2} = \frac{\pi}{a^n(a^2-1)}}$$

③ $f(x) = \cos(kx)/(x^2+a^2) \rightarrow f(z) = e^{ikz}/(z^2+a^2)$



Now $(z^2+a^2)^{-1} \rightarrow 0$ as $|z| = R \rightarrow \infty$

$\rightarrow$ integral over semi-circle $\rightarrow 0$ as $R \rightarrow \infty$ (Jordan's lemma)

$\Rightarrow \int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \times$ residues $\rightarrow$ poles at $z = \pm ia$

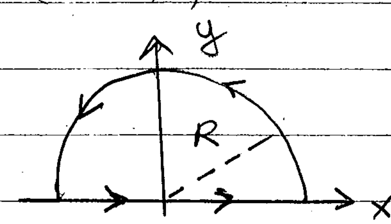
$\rightarrow$ only pole at $z = +ia$ is inside the contour

$\Rightarrow 2\pi i \times \text{residue} = (2\pi i)\, e^{-ka}(z-ia)/(z^2+a^2)\big|_{z=ia} = (2\pi i)\, e^{-ka}/2ia$

$$\Rightarrow \boxed{\int_{-\infty}^{\infty} \frac{\cos(kx)}{(x^2+a^2)}\,dx = \frac{\pi}{a} e^{-ka}}$$

Text 15.8 ④ $f(t) = t\sin(\alpha t)/(1+t^2) \rightarrow f(z) = -ize^{i\alpha z}/(1+z^2)$, $\alpha > 0$

Now $z/(1+z^2) \rightarrow 0$ as $|z| = R \rightarrow \infty$



$\rightarrow$ integral over semi-circle $\rightarrow 0$ as $R \rightarrow \infty$ (Jordan's lemma)

$\Rightarrow \int_{-\infty}^{\infty} \frac{t\sin(\alpha t)}{1+t^2}\,dt = 2\pi i \times$ residues $\rightarrow$ poles at $z = \pm i$

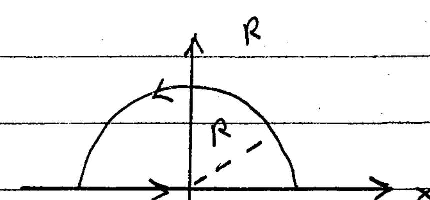
$\rightarrow$ only pole at $z = i$ is inside the contour

$\Rightarrow 2\pi i \times \text{residue} = (2\pi i)(-i)\, i\, e^{-\alpha}(z-i)/(1+z^2)\big|_{z=i}$

$= 2\pi i\, e^{-\alpha}/2i = \pi e^{-\alpha}$

$$\rightarrow \boxed{\int_{0}^{\infty} \frac{t\sin(\alpha t)}{1+t^2}\,dt = \frac{1}{2}\int_{-\infty}^{\infty} \frac{t\sin(\alpha t)}{1+t^2}\,dt = \frac{\pi}{2} e^{-\alpha}}$$

⑤ $f(x) = (x^2+a^2)^{-2} \rightarrow f(z) = (z^2+a^2)^{-2}$



Now $|z|(z^2+a^2)^{-2} \rightarrow 0$ as $|z| = R \rightarrow \infty$

$\rightarrow$ integral over semi-circle $\rightarrow 0$ as $R \rightarrow \infty$

$\Rightarrow \int_{0}^{\infty} \frac{dx}{(x^2+a^2)^{-2}} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)^{-2}} = \frac{1}{2}\, 2\pi i \times$ residues

(5) [continued] $\rightarrow$ double poles at $z = \pm ia$

$\rightarrow$ only pole a $z = ia$ is inside the contour

$$\Rightarrow \cancel{2\pi i \times \text{residue}} = (\pi i)\frac{d}{dz}\left[\frac{(z-ia)^2}{(z^2+a^2)^2}\right]\Bigg|_{z=ia} = (\pi i)\frac{d}{dz}\left[\frac{1}{(z+ia)^2}\right]_{z=ia}$$

$$(\pi i)\left[-2/(2ia)^3\right] = (-2\pi i)/(-8ia^3) = \pi/(4a^3)$$

$$\Rightarrow \boxed{\int_0^\infty \frac{dx}{(x^2+a^2)^2} = \frac{\pi}{4a^3}}$$

(6) Let $z = e^{i\theta} \Rightarrow a+\cos\theta = a + (z+z^{-1})/2 = (z^2+2az+1)/2z$

$dz = ie^{i\theta}d\theta = iz\,d\theta \Rightarrow d\theta = -idz/z$

$$\Rightarrow \int_0^\pi \frac{d\theta}{(a+\cos\theta)^2} = \frac{1}{2}\int_0^{2\pi}\frac{d\theta}{(a+\cos\theta)^2} = -2i\oint \frac{z\,dz}{(z^2+2az+1)^2}$$

double poles occur where $z^2+2az+1=0 \Rightarrow z = -a\pm\sqrt{a^2-1} \equiv z_\pm$

Since $a>1$ the pole at $z_- = -a-\sqrt{a^2-1}$ is outside the unit circle

At $z = -a+\sqrt{a^2-1} = z_+$,

$$\cancel{2\pi i \times \text{residue}} = 4\pi\,\frac{d}{dz}\left(\frac{z(z-z_+)^2}{[(z-z_+)(z-z_-)]^2}\right)\Bigg|_{z=z_+}$$

$$= 4\pi\left[\frac{1}{(z-z_-)^2} - \frac{2z}{(z-z_-)^3}\right]\Bigg|_{z=z_+} = \frac{4\pi}{(z_+-z_-)^3}\left[-(z_++z_-)\right]$$

$$= \frac{4\pi}{\left[2\sqrt{a^2-1}\right]^3}(2a) = \frac{\pi a}{(a^2-1)^{3/2}}$$

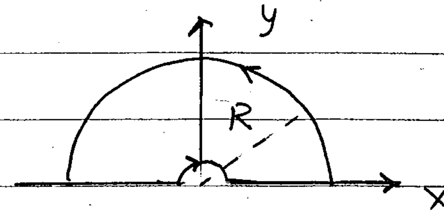
$$\Rightarrow \boxed{\int_0^\pi \frac{d\theta}{(a+\cos\theta)^2} = \frac{\pi a}{(a^2-1)^{3/2}}}$$

(7) $f(x) = [\cos(bx) - \cos(ax)]/x^2 \to f(z) = (e^{ibz} - e^{iaz})/z^2, \quad a > b > 0$

Now $z^{-2} \to 0$ as $R \to \infty$

→ integral over semi-circle → 0

as $R \to \infty$ (Jordan's lemma)



Integral over small semi-circle $= -\pi i \times$ residue at $z = 0$

double pole at $z = 0$

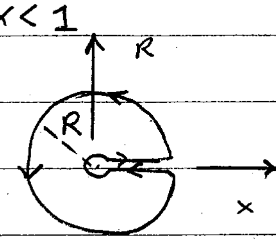
$\to$ ~~$-\pi i \times$ residue~~ $= -\pi i \dfrac{d}{dz}\left[z^2 f(z)\right]\Big|_{z=0} = -\pi i(ib - ia) = \pi(b-a)$

$$\Rightarrow \boxed{\int_{-\infty}^{\infty} \frac{\cos(bx) - \cos(ax)}{x^2}\,dx = \pi(a-b)} \quad \text{(no poles inside contour)}$$

Text 15.11] (8) $f(x) = x^{-\alpha}/1+x \to f(z) = z^{-\alpha}/(1+z)$ with $0 < \alpha < 1$

Fcn has a branch pt at $z = 0$

→ Choose contour with branch cut along the positive real axis



For $\alpha > 0$, $|z| z^{-\alpha}/(1+z) \to 0$ as $R \to \infty$

⇒ no contrib. on large semi-circle as $R \to \infty$

On the small semi-circle, $z = \rho e^{i\phi} \Rightarrow dz = i\rho e^{i\phi} d\phi = iz\, d\phi$

$\Rightarrow z^{-\alpha} dz = i z^{1-\alpha} d\phi \to 0$ as $\rho \to 0$ since $\alpha < 1$

⇒ no contrib on small semi-circle as $\rho \to 0$

On upper straight segment, $z = x$; on lower segment, $z = xe^{i2\pi}$

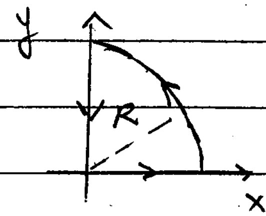
$\to z^{-\alpha} = x^{-\alpha} e^{-2\pi\alpha i}$

$\Rightarrow \oint_0^{\infty} \dfrac{z^{-\alpha}}{1+z} dz = (1 - e^{-2\pi\alpha i}) \displaystyle\int_0^{\infty} \dfrac{x^{-\alpha}}{1+x} dx =$ ~~$2\pi i \times$ residues~~

simple pole at $z = -1 \to$ residue $= (-1)^{-\alpha} = (e^{\pi i})^{-\alpha} = e^{-\pi i \alpha}$

⑧ [continued]

$$\Rightarrow \boxed{\int_0^\infty \frac{x^{-\alpha}}{1+x}\,dx = \frac{2\pi i\, e^{-\pi i\alpha}}{1-e^{-2\pi i\alpha}} = \frac{2\pi i}{e^{\pi i\alpha}-e^{-\pi i\alpha}} = \frac{\pi}{\sin(\pi\alpha)}}$$



⑨ $f(x) = (1+x^4)^{-1} \rightarrow f(z) = (1+z^4)^{-1}$

$|z|(1+z^4)^{-1} \rightarrow 0$ as $R\rightarrow\infty$

$\Rightarrow$ no contrib. on quarter-cricle as $R\rightarrow\infty$

On the real axis, $z=x$ and on the imag. axis $z = re^{i\pi/2}$.

$\Rightarrow dz = e^{i\pi/2}\,dr$ and $z^4 = (re^{i\pi/2})^4 = r^4$

$$\oint \frac{dz}{(1+z^4)^{-1}} = \left(1-e^{i\pi/2}\right)\int_0^\infty \frac{dx}{1+x^4} = \cancel{2\pi i \times \text{residues}}$$

poles at $z^4=-1 \Rightarrow z^2 = \pm i = e^{\pm i\pi/2} \Rightarrow z = \pm e^{\pm i\pi/4}$

$\rightarrow z = \pm(1\pm i)/\sqrt{2} \rightarrow$ define $z_1 = (1+i)/\sqrt{2}$, $z_2 = (1-i)/\sqrt{2}$,

$z_3 = (-1+i)/\sqrt{2}$, $z_4 = (-1-i)/\sqrt{2}$ $\Rightarrow$ only z_1 inside contour

$$\cancel{\Rightarrow 2\pi i\times\text{residue} = 2\pi i}\left(\frac{z-z_1}{1+z^4}\right)\Bigg|_{z=z_1} = \frac{2\pi i}{(z_1-z_2)(z_1-z_3)(z_1-z_4)}$$

$$= \frac{2\pi i}{(\sqrt{2}i)(\sqrt{2})[\sqrt{2}(i+1)]} = \frac{\pi}{\sqrt{2}}\left(\frac{1}{i+1}\right) = \frac{\pi(1-i)}{2\sqrt{2}}$$

$$\Rightarrow \boxed{\int_0^\infty \frac{dx}{1+x^4} = \frac{\pi}{2\sqrt{2}}\left(\frac{1-i}{1-e^{i\pi/2}}\right) = \frac{\pi}{2\sqrt{2}}}$$