PHZ 3113 Solutions

Test I – 10 Feb 2017

1 (a) $\vec{A} = 2xz\,\hat{x} + 2yz^2\,\hat{y} + (x^2 + 2y^2z - 1)\,\hat{z}$

6/ $\vec{\nabla}\times\vec{A} = (\partial A_z/\partial y - \partial A_y/\partial z)\,\hat{x} + (\partial A_x/\partial z - \partial A_z/\partial x)\,\hat{y}$

$+ (\partial A_y/\partial x - \partial A_x/\partial y)\,\hat{z}$

$= (4yz - 4yz)\,\hat{x} + (2x - 2x)\,\hat{y} + (0 - 0)\,\hat{z} \Rightarrow \boxed{\vec{\nabla}\times\vec{A} = 0}$

(b) $\dfrac{\partial\Phi}{\partial x} = A_x = 2xz \Rightarrow \Phi = x^2z + f(y,z)$

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$\partial\Phi/\partial y = A_y \Rightarrow \partial f/\partial y = 2yz^2 \Rightarrow f = y^2z^2 + g(z)$

$\Rightarrow \Phi = x^2z + y^2z^2 + g(z)$

$\dfrac{\partial\Phi}{\partial z} = A_z \Rightarrow x^2 + 2y^2z + \dfrac{dg}{dz} = x^2 + 2y^2z - 1 \Rightarrow \dfrac{dg}{dz} = -1$

$\Rightarrow g = -z + c \quad \Rightarrow \boxed{\Phi = x^2z + y^2z^2 - z + c}$

2 (a) $\vec{A} = \tfrac{1}{3}\mu_0\sigma\omega a r\sin\theta\,\hat{\phi}$

8/ $\vec{\nabla}\times\vec{A} = \dfrac{1}{r\sin\theta}\left(\dfrac{\partial}{\partial\theta}(\sin\theta A_\phi) - \dfrac{\partial A_\theta}{\partial\phi}\right)\hat{r} + \dfrac{1}{r}\left[\dfrac{1}{\sin\theta}\dfrac{\partial A_r}{\partial\phi} - \dfrac{\partial}{\partial r}(rA_\phi)\right]\hat{\theta}$

$+ \tfrac{1}{r}\left[\partial/\partial r(rA_\theta) - \partial A_r/\partial\theta\right]\hat{\phi}$

$= \tfrac{1}{3}\mu_0\sigma\omega a\left[2\cos\theta\,\hat{r} - 2\sin\theta\,\hat{\theta}\right]$

Now $\hat{r}\cos\theta - \hat{\theta}\sin\theta = \hat{z} \Rightarrow \boxed{\vec{B} = \tfrac{2}{3}\mu_0\sigma\omega a\,\hat{z}}$

(b) $\vec{A} = \tfrac{1}{3}\mu_0\sigma\omega\dfrac{a^4}{r^2}\sin\theta\,\hat{\phi}$

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$\Rightarrow \vec{\nabla}\times\vec{A} = \tfrac{1}{3}\mu_0\sigma\omega a^4\left[\dfrac{2\cos\theta}{r^3}\hat{r} + \dfrac{\sin\theta}{r^3}\hat{\theta}\right]$

$\Rightarrow \boxed{\vec{B} = \tfrac{1}{3}\mu_0\sigma\omega\dfrac{a^4}{r^3}(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta})}$

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Solutions

(3) (a) $\vec{A} = r\cos\theta\,\hat{r} + r\sin\theta\,\hat{\theta} + r\sin\theta\cos\phi\,\hat{\phi}$

4/ $\vec{\nabla}\cdot\vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$

$= 3\cos\theta + 2\cos\theta - \sin\phi \Rightarrow \boxed{\vec{\nabla}\cdot\vec{A} = 5\cos\theta - \sin\phi}$

(b) $\int \vec{\nabla}\cdot\vec{A}\,d\tau = \int_0^R r^2 dr \int_0^1 d(\cos\theta) \int_0^{2\pi} d\phi\,(5\cos\theta - \sin\phi)$

5/ $= \frac{1}{3}R^3 \int_0^1 d(\cos\theta)\left[(10\pi)\cos\theta + 0\right] = \frac{5}{3}\pi R^3 \Rightarrow \boxed{\int \vec{\nabla}\cdot\vec{A}\,d\tau = \frac{5}{3}\pi R^3}$

(c) $\int \vec{A}\cdot\hat{n}\,dS = R^2 \int_0^1 d(\cos\theta) \int_0^{2\pi} d\phi\, A_r(r=R) + \int_0^R r^2 dr \int_0^{2\pi} d\phi\, A_\theta(\theta = \frac{\pi}{2})$

6/ $= 2\pi R^3\left(\frac{1}{2}\right) + \frac{1}{3}R^3(2\pi) = \frac{5}{3}\pi R^3 \Rightarrow \boxed{\int \vec{A}\cdot\hat{n}\,dS = \frac{5}{3}\pi R^3}$

(4) (a) $\vec{B} = \vec{\nabla}f \times \vec{\nabla}g \Rightarrow \vec{\nabla}\cdot\vec{B} = \vec{\nabla}g\cdot(\vec{\nabla}\times\vec{\nabla}f) - \vec{\nabla}f\cdot(\vec{\nabla}\times\vec{\nabla}g)$

3 But $\vec{\nabla}\times\vec{\nabla}f = \vec{\nabla}\times\vec{\nabla}g = 0 \Rightarrow \boxed{\vec{\nabla}\cdot\vec{B} = 0}$

(b) $A = \frac{1}{2}(f\vec{\nabla}g - g\vec{\nabla}f) \Rightarrow \vec{\nabla}\times\vec{A} = \frac{1}{2}(\vec{\nabla}f\times\vec{\nabla}g + f\vec{\nabla}\times\vec{\nabla}g - \vec{\nabla}g\times\vec{\nabla}f - g\vec{\nabla}\times\vec{\nabla}f$

5/ But $\vec{\nabla}\times\vec{\nabla}g = \vec{\nabla}\times\vec{\nabla}f = 0$ and $-\vec{\nabla}g\times\vec{\nabla}f = \vec{\nabla}f\times\vec{\nabla}g$

$\Rightarrow \boxed{\vec{\nabla}\times\vec{A} = \vec{\nabla}f\times\vec{\nabla}g = \vec{B}}$

7/ (c) $\vec{\nabla}\times(f\vec{A}) = \vec{\nabla}f\times\vec{A} + f\vec{\nabla}\times\vec{A} = 0 \Rightarrow f\vec{\nabla}\times\vec{A} = -\vec{\nabla}f\times\vec{A}$

$\Rightarrow \vec{A}\cdot(f\vec{\nabla}\times\vec{A}) = f\,\vec{A}\cdot\vec{\nabla}\times\vec{A} = -\vec{A}\cdot(\vec{\nabla}f\times\vec{A}) = 0$

But $f \neq 0 \Rightarrow \vec{A}\cdot\vec{\nabla}\times\vec{A} = 0 \Rightarrow \boxed{\vec{A} \perp \vec{\nabla}\times\vec{A}}$