PHZ 3113 — Solutions —

Test II – 10 March 2017

1 (a) $\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix} = \begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}0&-i\\i&0\end{pmatrix} = \begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix} = \begin{pmatrix}1&0\\0&1\end{pmatrix}$

6 $\Rightarrow \underline{S_x^2 = S_y^2 = S_z^2 = \mathbb{I}}$

$\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}0&-i\\i&0\end{pmatrix} = -\begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix} = \begin{pmatrix}i&0\\0&-i\end{pmatrix} \Rightarrow \underline{S_xS_y = -S_yS_x = iS_z}$

$\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix} = -\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix} = \begin{pmatrix}0&1\\-1&0\end{pmatrix} = iS_y \Rightarrow \underline{S_zS_x = -S_xS_z = iS_y}$

$\begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix} = -\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}0&-i\\i&0\end{pmatrix} = \begin{pmatrix}0&i\\i&0\end{pmatrix} \Rightarrow \underline{S_yS_z = -S_zS_y = iS_x}$

(b) $S(\vec{a})\cdot S(\vec{b}) = (a_xS_x + a_yS_y + a_zS_z)(b_xS_x + b_yS_y + b_zS_z)$

5 $= a_xb_xS_x^2 + a_yb_yS_y^2 + a_zb_zS_z^2 + (a_xb_y - a_yb_x)S_xS_y + (a_zb_x - a_xb_z)S_zS_x$

$+ (a_yb_z - a_zb_y)S_yS_z = (a_xb_x + a_yb_y + a_zb_z)\,\mathbb{I} + i(a_xb_y - a_yb_x)S_z$

$+ i(a_zb_x - a_xb_z)S_y + i(a_yb_z - a_zb_y)S_x$

Now $a_xb_x + a_yb_y + a_zb_z = \vec{a}\cdot\vec{b}$, $(a_xb_y - a_yb_x) = (\vec{a}\times\vec{b})_z$,

$a_zb_x - a_xb_z = (\vec{a}\times\vec{b})_y$ and $(a_yb_z - a_zb_y) = (\vec{a}\times\vec{b})_x$

$\Rightarrow \boxed{S(\vec{a})\cdot S(\vec{b}) = \vec{a}\cdot\vec{b}\,\mathbb{I} + i(\vec{a}\times\vec{b})\cdot\vec{S}}$

4 (c) $S(\vec{b})\cdot S(\vec{a}) = (\vec{b}\cdot\vec{a})\mathbb{I} + i(\vec{b}\times\vec{a})\cdot\vec{S} = (\vec{a}\cdot\vec{b})\mathbb{I} - i(\vec{a}\times\vec{b})\cdot\vec{S}$

so $S(\vec{b})\cdot S(\vec{a}) = S(\vec{a})\cdot S(\vec{b}) \Rightarrow \vec{a}\times\vec{b} = 0 \Rightarrow \underline{\vec{a} \parallel \vec{b}}$

2 (a) $A\cdot\vec{x} = \vec{b}$ with $A = \begin{pmatrix}2&3&1\\1&1&1\\5&-1&10\end{pmatrix}$ and $\vec{b} = \begin{pmatrix}11\\6\\34\end{pmatrix}$

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$\rightarrow \det A = 2(11) - 3(5) + 1(-6) = 22 - 15 - 6 = 1$

$\Rightarrow \boxed{\det A = 1} \neq 0 \Rightarrow$ unique soln

(b) A^{-1} has elements $(A^{-1})_{ij} = \dfrac{C_{ji}}{\det(A)}$

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$\rightarrow$ need cofactors of matrix A

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(2) (b) (continued) The cofactors are

$C_{11} = 11$, $C_{12} = -5$, $C_{13} = -6$, $C_{21} = -31$, $C_{22} = 15$

$C_{23} = 17$, $C_{31} = 2$, $C_{32} = -1$, $C_{33} = -1$

$$\Rightarrow \boxed{A^{-1} = \begin{pmatrix} 11 & -31 & 2 \\ -5 & 15 & -1 \\ -6 & 17 & -1 \end{pmatrix}} \qquad \text{check } A \cdot A^{-1} = I$$

(c)

4/ $$\vec{x} = A^{-1}\vec{b} = \begin{pmatrix} 11 & -31 & 2 \\ -5 & 15 & -1 \\ -6 & 17 & -1 \end{pmatrix}\begin{pmatrix} 11 \\ 6 \\ 34 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} \Rightarrow \boxed{\begin{matrix} x_1 = 3 \\ x_2 = 1 \\ x_3 = 2 \end{matrix}}$$

(3) (a) $\det(\lambda I - A) = 0 \Rightarrow (\lambda-1)\left[(\lambda-1)^2 - 8\right] - \sqrt{8}\,\sqrt{8}(\lambda-1) = 0$

4/ $\Rightarrow (\lambda-1)\left[(\lambda-1)^2 - 16\right] = (\lambda-1)(\lambda^2 - 2\lambda - 15) = (\lambda-1)(\lambda+3)(\lambda-5) = 0$

$\Rightarrow$ $\boxed{\text{eigenvalues are } \lambda = 1, -3, 5}$

(b) Eqs for eigenvectors are

8/ $x_1 + \sqrt{8}\,x_2 = \lambda x_1$ (1) $\quad \sqrt{8}\,x_1 + x_2 + \sqrt{8}\,x_3 = \lambda x_2$ (2) $\quad \sqrt{8}\,x_2 + x_3 = \lambda x_3$ (3)

Eq (1) − Eq (3) $\Rightarrow x_1 - x_3 = \lambda(x_1 - x_3) \Rightarrow \underline{x_3 = x_1 \text{ if } \lambda \neq 1}$

From eq (1), $\underline{x_2 = (\lambda - 1)x_1/\sqrt{8}} = -\sqrt{2}\,x_1$ $(\lambda = -3)$ or $+\sqrt{2}\,x_1$ $(\lambda = 5)$

If $\lambda = 1$, then $\underline{x_2 = 0}$ and from equ (2) $\underline{x_3 = -x_1}$

so normalized eigenvectors are

$\boxed{\lambda = 1: \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}}$ $\quad \boxed{\lambda = -3: \frac{1}{2}\begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}}$ $\quad \boxed{\lambda = 5: \frac{1}{2}\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}}$

(c) diagonalized matrix $= RAR^T$

3/

$$\text{with } R^T = \begin{pmatrix} 1/\sqrt{2} & 1/2 & 1/2 \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/2 & 1/2 \end{pmatrix} \quad \text{(columns are normalized eigenvectors)}$$

$$\text{check } R \cdot A \cdot R^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

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(4) (a) $73x_1^2 + 72x_1x_2 + 52x_2^2 = 100$

4/ $\Rightarrow a_{11} = 73,\quad a_{22} = 52,\quad a_{12} = a_{21} = \tfrac{1}{2}(72) = 36$

$$\Rightarrow \boxed{A = \begin{pmatrix} 73 & 36 \\ 36 & 52 \end{pmatrix}}$$

(b) $\det(\lambda I - A) = 0 \Rightarrow (\lambda - 73)(\lambda - 52) - 36^2 = 0$

1/ $\Rightarrow \lambda^2 - 125\lambda + 2500 = 0 \Rightarrow \lambda = \tfrac{1}{2}\left[125 \pm \sqrt{125^2 - 10000}\right]$

$\Rightarrow$ eigenvalues are $\lambda = \tfrac{1}{2}(125 \pm 75) \Rightarrow \boxed{\lambda = 25,\ 100}$

Now choose x_1' = coord in direction of $\lambda = 25$ eigenvector

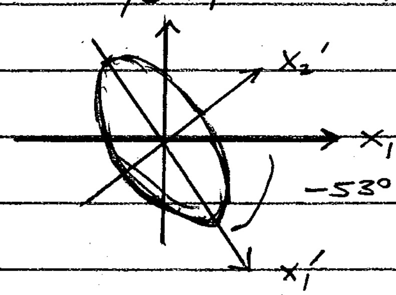
x_2' = coord " $\lambda = 100$ eigenvector

$\Rightarrow 25x_1'^2 + 100x_2'^2 = 100 \Rightarrow x_1'^2/4 + x_2'^2 = 1$

$$\boxed{\begin{array}{l} \text{major axis length} = 2\sqrt{4} = 4 \\ \text{minor axis length} = 2\sqrt{1} = 2 \end{array}}$$

(c) Major axis coords satisfy equ $73x_1 + 36x_2 = 25x_1$

4/ $\Rightarrow x_2 = -\tfrac{4}{3}x_1 \Rightarrow \tan\phi = -\tfrac{4}{3} \Rightarrow \boxed{\phi = -53^\circ}$



($53°$ below x_1 axis)