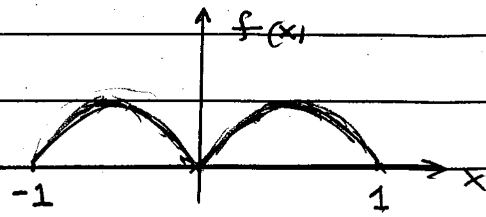
PHZ 3113
Test III
7 April 2017

— Solutions —



1 (a) 4/

$$f(x) = 4x(x+1), \quad -1 \le x \le 0$$
$$\quad = 4x(x-1), \quad 0 \le x \le 1$$

symmetric → no sine terms

$$\underline{a_0} = \int_{-1}^{1} f(x)dx = 2\int_0^1 f(x)dx = 8\int_0^1 x(x-1)\,dx = \frac{8}{3} - \frac{8}{2} = \underline{-\frac{4}{3}}$$

$$\underline{a_n} = 2\int_0^1 f(x)\cos(n\pi x)\,dx = 8\int_0^1 x(x-1)\cos(n\pi x)dx$$

$$= (8/n\pi)\left[\underbrace{x(x-1)\sin(n\pi x)\Big|_0^1}_{0} - \int_0^1 (2x-1)\sin(n\pi x)dx\right]$$

$$= \frac{8}{(n\pi)^2}\left[(2x-1)\cos(n\pi x)\Big|_0^1 - 2\underbrace{\int_0^1 \cos(n\pi x)dx}_{0}\right] = \frac{8}{(n\pi)^2}\left[\cos(n\pi) + 1\right]$$

$$= \underline{16/(n\pi)^2, \; n \text{ even}}; \quad = 0, \; n \text{ odd}$$

$$\boxed{f(x) = -\frac{2}{3} + \frac{16}{\pi^2}\sum_{n\,\text{even} \ge 2} \frac{\cos(n\pi x)}{n^2}}$$

(b) 4/ For $x = 0$, $f(x) = 0$ and $\cos(n\pi x) = 1$

$$\Rightarrow \boxed{\sum_{n\,\text{even} \ge 2} \frac{1}{n^2} = \frac{2}{3}\,\frac{\pi^2}{16} = \frac{\pi^2}{24}}$$

2 (a) 8/

$$f(t) = e^{-\gamma t}\sin\beta t = e^{-\gamma t}(e^{i\beta t} - e^{-i\beta t})/2i, \quad \text{for } t \ge 0$$

$$\Rightarrow \tilde{f}(\omega) = \frac{1}{\sqrt{2\pi}}\int_0^\infty f(t)e^{-i\omega t}dt = \frac{1}{\sqrt{2\pi}}\,\frac{1}{2i}\int_0^\infty \left[e^{[-\gamma + i(\beta-\omega)]t} - e^{[-\gamma - i(\beta+\omega)]t}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\,\frac{1}{2i}\left[\frac{e^{[-\gamma+i(\beta-\omega)]t}}{-\gamma+i(\beta-\omega)} - \frac{e^{[-\gamma-i(\beta+\omega)]t}}{-\gamma-i(\beta+\omega)}\right]_0^\infty$$

$$= \frac{1}{\sqrt{2\pi}}\,\frac{1}{2i}\,\frac{-\gamma+i(\beta-\omega) - [-\gamma-i(\beta+\omega)]}{[-\gamma+i(\beta-\omega)][-\gamma-i(\beta+\omega)]} = \frac{\beta}{\sqrt{2\pi}}\,\frac{1}{(\gamma+i\omega)^2+\beta^2}$$

$$\Rightarrow \boxed{\tilde{f}(\omega) = \frac{1}{\sqrt{2\pi}}\,\frac{\beta}{(\gamma+i\omega)^2+\beta^2}}$$

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— Solns, p. 2 —

(2)(b) $I(t) = \int_{-\infty}^{\infty} f(t-u)\, V(u)\, du$

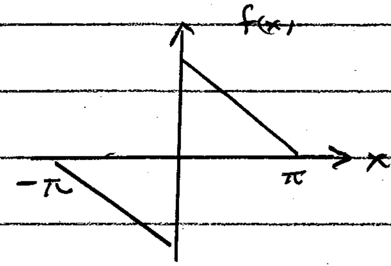
4/ $\Rightarrow Q = \int_{-\infty}^{\infty} I(t)\,dt = \int_{-\infty}^{\infty}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\tilde{I}(\omega)\, e^{i\omega t}\,d\omega\right] dt$

$= \int_{-\infty}^{\infty} \tilde{I}(\omega)\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\omega t}\,dt\right] d\omega = \int_{-\infty}^{\infty}\tilde{I}(\omega)\sqrt{2\pi}\,\delta(\omega)\,d\omega$

$\Rightarrow \boxed{Q = \sqrt{2\pi}\;\tilde{I}(0)}$

(c) Convolution thm $\Rightarrow \tilde{I}(\omega) = \sqrt{2\pi}\,\tilde{f}(\omega)\,\tilde{V}(\omega)$

3/ $\Rightarrow \boxed{Q = 2\pi\,\tilde{f}(0)\,\tilde{V}(0) = \sqrt{2\pi}\,\frac{\beta\tilde{V}(0)}{\gamma^2+\beta^2}}$



(3)(a) $f(x) = -\tfrac{1}{2}(x+\pi), \quad -\pi \le x < 0$

8/ $\quad\;\; = \tfrac{1}{2}(\pi - x), \quad 0 < x \le \pi$

antisymmetric $\rightarrow$ no constant or cosine terms

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx = \frac{2}{\pi}\int_0^{\pi}\frac{1}{2}(\pi-x)\sin(nx)\,dx$

$= \frac{1}{n\pi}\left[-(\pi-x)\cos(nx)\Big|_0^{\pi} - \int_0^{\pi}\underbrace{\cos(nx)\,dx}_{0}\right] = \frac{1}{n} \Rightarrow \boxed{f(x) = \sum_{n>0}\frac{\sin(nx)}{n}}$

(b) For $x > 0$,

4/ $\int f(x)\,dx = \frac{\pi}{2}x - \frac{1}{4}x^2 + C_1 = -\frac{1}{4}(\pi-x)^2 + \frac{\pi^2}{4} + C_1$

$\int [\sin(nx)/n]\,dx = -\cos(nx)/n^2 + C_2$

$\Rightarrow \boxed{\frac{1}{4}(\pi-x)^2 = \sum_{n>0}\frac{\cos(nx)}{n^2} + C}$ where $C = \frac{\pi^2}{4} + C_1 - C_2$

(c) For $x = \pi$, $(\pi - x)^2 = 0$ and $\cos(nx) = \cos(n\pi) = (-1)^n$

3/ $\Rightarrow \sum_{n>0}\frac{\cos(nx)}{n^2} = \sum_{n>0}\frac{(-1)^n}{n^2} = -\frac{\pi^2}{12} \Rightarrow \boxed{C = \frac{\pi^2}{12}}$

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4 (a) For $f(\vec{r}) = f(r)$, 3d F.T. given by

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$$F(\vec{\alpha}) = \frac{1}{(2\pi)^{3/2}} \int f(r)\, e^{-i\vec{\alpha}\cdot\vec{r}}\, d^3r$$

Use spherical coords with $\hat{z}$ along $\vec{\alpha}$ $\Rightarrow$ $\vec{\alpha}\cdot\vec{r} = \alpha r\cos\theta$
and $d^3r = r^2 dr\, d(\cos\theta)\, d\phi$

$$\Rightarrow F(\vec{\alpha}) = \frac{1}{(2\pi)^{3/2}} \int_0^{2\pi} d\phi \int_0^\infty r^2 f(r) \left[\int_{-1}^{1} e^{-i\alpha r\cos\theta}\, d(\cos\theta)\right] dr$$

$$\int_0^{2\pi} d\phi = 2\pi \qquad \int_{-1}^{1} e^{-i\alpha r\cos\theta}\, d(\cos\theta) = \frac{e^{-i\alpha r} - e^{i\alpha r}}{-i\alpha r} = \frac{2}{\alpha r}\sin(\alpha r)$$

$$\Rightarrow \boxed{F(\vec{\alpha}) = F(\alpha) = \sqrt{\frac{2}{\pi}}\,\frac{1}{\alpha}\int_0^\infty f(r)\, r\sin(\alpha r)\, dr}$$

(b) $f(r) = e^{-\mu r}/r$

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$$\rightarrow F(\alpha) = \sqrt{\frac{2}{\pi}}\,\frac{1}{\alpha}\int_0^\infty e^{-\mu r}\sin(\alpha r)\, dr = \sqrt{\frac{2}{\pi}}\,\frac{1}{2i\alpha}\int_0^\infty \left[e^{(-\mu+i\alpha)r} - e^{(-\mu-i\alpha)r}\right] dr$$

$$= \sqrt{\frac{2}{\pi}}\,\frac{1}{2i\alpha}\left[\frac{1}{-\mu-i\alpha} - \frac{1}{-\mu+i\alpha}\right] = \sqrt{\frac{2}{\pi}}\,\frac{1}{2i\alpha}\,\frac{-\mu+i\alpha-(-\mu-i\alpha)}{\mu^2+\alpha^2}$$

$$\rightarrow \boxed{F(\alpha) = \sqrt{\frac{2}{\pi}}\,\frac{1}{\mu^2+\alpha^2}}$$