PHZ 3113

Test IV – 24 April 2017

— Solutions —

(1) (a) $u = xy \Rightarrow \partial u/\partial x = y$ and $\partial u/\partial y = x$

15/ CR relations $\rightarrow \partial v/\partial y = \partial u/\partial x = y \Rightarrow v = \tfrac{1}{2}y^2 + f(x)$

$\partial v/\partial x = -\partial u/\partial y \Rightarrow f'(x) = -x \Rightarrow f(x) = -\tfrac{1}{2}x^2 + c$

Set $c = 0 \Rightarrow \boxed{v = \tfrac{1}{2}(y^2 - x^2)} \Rightarrow f(z) = xy + i/2\,(y^2 - x^2)$

$\rightarrow \boxed{f(z) = -i/2\,(x+iy)^2 = -i/2\; z^2}$

(b) $v = 3x^2y - y^3 \Rightarrow \partial v/\partial x = 6xy$ and $\partial v/\partial y = 3(x^2 - y^2)$

CR relations $\rightarrow \partial u/\partial y = -\partial v/\partial x \Rightarrow u = -3xy^2 + f(x)$

$\partial u/\partial x = \partial v/\partial y \Rightarrow -3y^2 + f'(x) = 3(x^2 - y^2) \Rightarrow f'(x) = 3x^2$

$\Rightarrow f(x) = x^3 + C \rightarrow$ Set $c = 0 \Rightarrow \boxed{u = x^3 - 3xy^2}$

$\rightarrow f(z) = x^3 - 3xy^2 + i(3x^2y - y^3) = (x+iy)^3 \rightarrow \boxed{f(z) = z^3}$

(2) (a) $w = (z-1)/(z+1) = (x+iy-1)/(x+iy+1)$

$\Rightarrow w = (x-1+iy)(x+1-iy)/[(x+1)^2 + y^2]$

8/ $= \dfrac{x^2 - 1 + y^2 + 2iy}{(x+1)^2 + y^2} \Rightarrow \boxed{u = \dfrac{x^2 + y^2 - 1}{(x+1)^2 + y^2}}$ $\boxed{v = \dfrac{2y}{(x+1)^2 + y^2}}$

(b) imaginary axis in the z-plane $\rightarrow x = 0$

$\Rightarrow u^2 + v^2 = \dfrac{(y^2-1)^2}{(y^2+1)^2} + \dfrac{4y^2}{(y^2+1)^2} = \dfrac{y^4 + 2y^2 + 1}{(y^2+1)^2} = 1 \rightarrow$ unit circle

(c) $w = (z-i)/(z+i) = [x + i(y-1)]/[x + i(y+1)]$

$\rightarrow w = [x + i(y-1)][x - i(y+1)]/[x^2 + (y+1)^2]$

7/ $= \dfrac{x^2 + y^2 - 1 - 2ix}{x^2 + (y+1)^2} \Rightarrow \boxed{u = \dfrac{x^2 + y^2 - 1}{x^2 + (y+1)^2}}$ $\boxed{v = \dfrac{-2x}{x^2 + (y+1)^2}}$

(d) real axis in the z-plane $\rightarrow y = 0$

$\Rightarrow u^2 + v^2 = \dfrac{(x^2-1)^2}{(x^2+1)^2} + \dfrac{4x^2}{(x^2+1)^2} = \dfrac{x^4 + 2x^2 + 1}{(x^2+1)^2} = 1 \rightarrow$ unit circle

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(3) (a) $f(x) = \sin^2 x / x^2 = [1-\cos(2x)]/2x^2 \rightarrow f(z) = (1-e^{2iz})/2z^2$

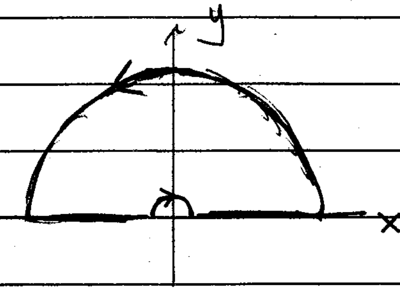
1/ $\rightarrow$ double pole at $z=0$

$\rightarrow$ residue $= d/dz(z^2 f(z))\big|_{z=0} = -\tfrac{1}{2}(2i)e^{2iz}\big|_{z=0} = -i$

$\rightarrow$ $\boxed{\text{residue} = -i}$

(b) $|z f(z)| \propto 1/R \rightarrow 0$ as $R \rightarrow \infty$

4/ so integral on large semi-circle vanishes



(c) Integral on small semi-circle

4/ $= -\pi i \times$ residue at $z=0 = -\pi$

no poles inside contour $\Rightarrow \oint f(z)\,dz = 0$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2}\,dx - \pi = 0 \Rightarrow \boxed{\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2}\,dx = \pi}$$

(4) (a) $f(x) = x^2/(x^4+1) \rightarrow f(z) = z^2/(z^4+1)$

4/ poles are at $z^4 = -1 \Rightarrow z^2 = \pm i = e^{\pm i\pi/2}$

$\Rightarrow z = \pm\sqrt{\pm i} = \pm e^{\pm i\pi/4} = \pm 1/\sqrt{2}\,(1 \pm i)$

$\boxed{\text{poles at } z_1 = 1/\sqrt{2}(1+i),\ z_2 = 1/\sqrt{2}(1-i),\ z_3 = -1/\sqrt{2}(1+i),\ z_4 = -1/\sqrt{2}(1-i)}$

(b) Poles at z_1 and z_4 are in upper half plane

6/ At $z = z_1$, residue $= \dfrac{z_1^2}{(z_1-z_2)(z_1-z_3)(z_1-z_4)} = \dfrac{i}{(\sqrt{2})^3}\,\dfrac{1}{(i)(1+i)(1)} = \dfrac{1}{2^{3/2}(1+i)}$

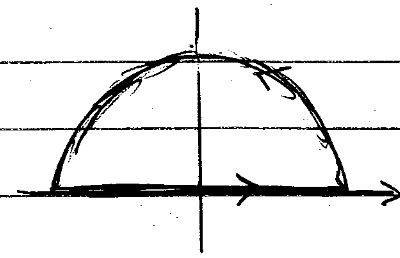
$\Rightarrow \boxed{R_1 = 2^{-3/2}(1+i)^{-1}}$

At $z = z_4$ residue $= \dfrac{z_4^2}{(z_4-z_1)(z_4-z_2)(z_4-z_3)} = \dfrac{-i}{(\sqrt{2})^3}\,\dfrac{1}{(-1)(i-1)\,i} = \dfrac{1}{2^{3/2}(i-1)}$

$\rightarrow \boxed{R_4 = 2^{-3/2}(i-1)^{-1}}$

4 c $|z\,f(z)| \propto 1/R^2 \to 0$ as $R \to \infty$

6/ so integral on semi-circle vanishes



$$\Rightarrow \int_{-\infty}^{\infty} \frac{x^2}{x^4+1}\,dx = 2\pi i(R_1+R_4) = \frac{2\pi i}{2^{3/2}}\left(\frac{1}{i+1}+\frac{1}{i-1}\right) = \frac{(2\pi i)(2i)}{2^{3/2}(-2)}$$

$$\rightarrow \boxed{\int_{-\infty}^{\infty} \frac{x^2}{x^4+1}\,dx = \frac{\pi}{\sqrt{2}}}$$