

Probability

3.1 Sample Spaces and Tree Diagrams

This section introduces terminology and some techniques which will eventually lead us to the basic concept of the probability of an event.

The **Rare Event Rule** for Inferential Statistics

If under a given assumption, the probability of a particular observed event is extremely small, we conclude that the assumption is probably not correct.

Here is an example of the Rare Event Rule: Say that you assume that a college graduate will have a starting salary of 75k or more, but a random survey of 32 recent graduates indicates that the starting salaries were around 35k. If your assumption is actually true the probability that a sample of 32 recent grads would have an average salary of only 35k would be extremely small, so we should conclude your assumption was wrong (in actuality, we need to know what the standard deviation is before we could decide how probable the above sample results would be, but we will get to that later).

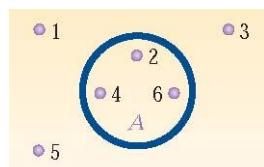
Before we get to probability, there are some terms we need to discuss:

A **random sample** is selected in such a way that every different sample of size n has an equal chance of selection.

An **experiment** is an act of observation that leads to a single outcome that cannot be predicted with certainty.

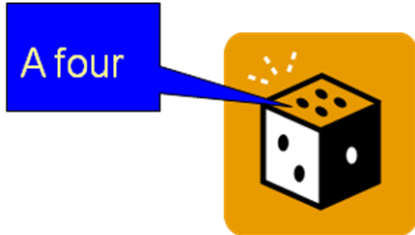
An **event** is a specific collection of sample points.

For example: Event A: Observe an even number.



A **sample point (or simple event)** is the most basic outcome of an experiment. For example, rolling a four on a single roll of a die, is a simple event.

Sample point ~ outcome



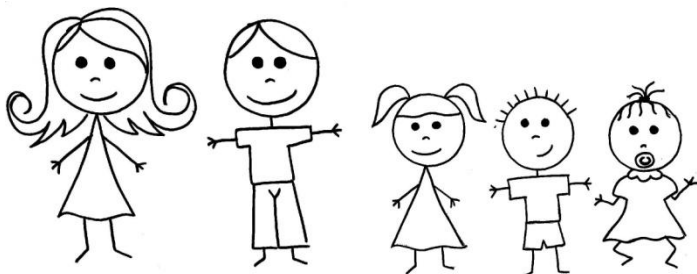
A **sample space** (denoted S) is the collection of all possible outcomes of an experiment.

For example: a roll of a single die has the sample space:

$$S: \{1, 2, 3, 4, 5, 6\}$$

Sometimes it is hard to list all of the possible outcomes for an experiment, but there are some helpful techniques available. One of these techniques is the Tree Diagram.

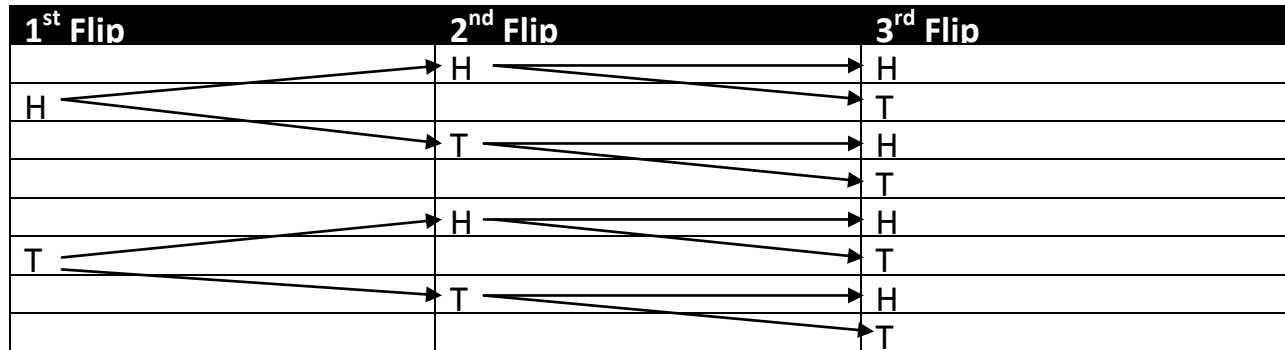
Example 28: List the different possible families that can occur when a couple has three children...



Example 29: List the possible outcomes for three flips of a fair coin...



Solution: HHH, HHT, HTH, HTT, THH, THT, TTH, TTT



3.2 Basic Probability

Here are some commonly used notational conventions:

A, B, C, \dots denote specific events.

$P(A)$ denotes the probability of event A occurring.

The **probability of an event A** is calculated by summing the probabilities of the sample points in the sample space for A . In other words:

$$P(A) = \frac{n}{N},$$
 $n = \# \text{ of times you observed event } A \text{ (number of ways } A \text{ can happen), } N = \text{number of observations (Number of total possibilities).}$

You might have noticed that the statements in the parenthesis in the above definition seem to define a second definition of probability. That is because there are two ways to think about probability. Let's consider each separately:

Rule 1: Relative Frequency Approximation of Probability

Conduct (or observe) a procedure, and count the number of times event A actually occurs. Based on these actual results, $P(A)$ is estimated as follows:

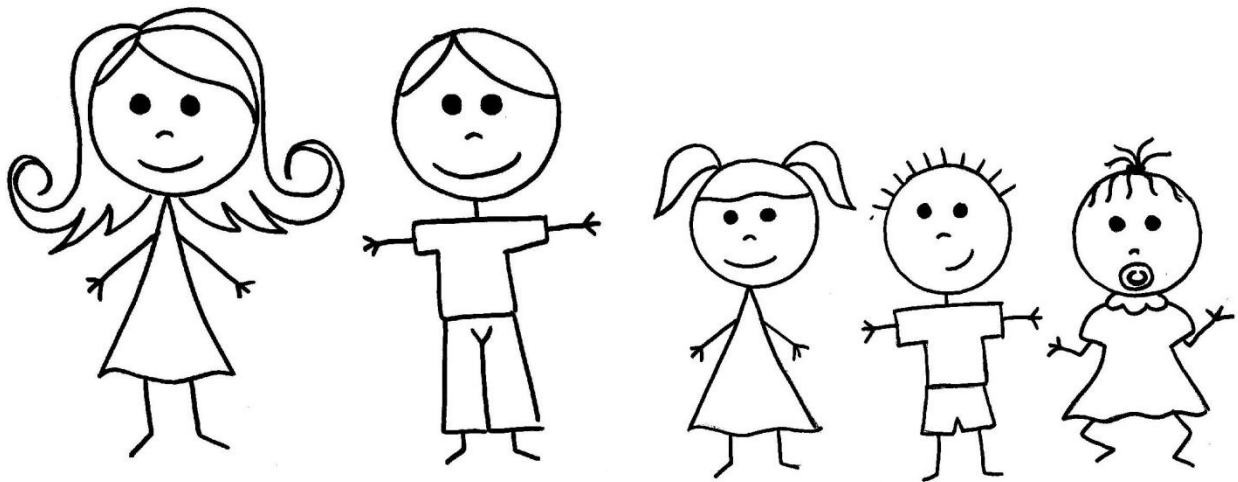
$$P(A) = \frac{\text{number of times } A \text{ occurred}}{\text{number of times trial was repeated}}$$

Rule 2: Classical Approach to Probability (Requires Equally Likely Outcomes)

Assume that a given procedure has n different simple events and that each of those simple events has an equal chance of occurring. If event A can occur in s of these n ways, then

$$P(A) = \frac{\text{number of ways } A \text{ can occur}}{\text{number of total possible outcomes}}$$

Example 30: Using the sample space created above for having three children, find the probability of having at least two girls.



Here are some things to keep in mind when working with probability.

Law of Large Numbers: As a procedure is repeated over and over again, the relative frequency of an event tends to approach the true probability for that event.

Probability Rules for Sample Points:

1. All sample point probabilities must be between 0 and 1. **That is, $0 \leq P(A) \leq 1$.** (0 indicates an impossible outcome and 1 a certain outcome.)
2. The sum of the individual sample point probabilities must be equal to 1. **That is, $\sum_{i=1}^n p_i = 1$.**
 (if you have listed all of the possible outcomes of an experiment then you must have 100% of the probability.)

Example 31: Use the data summarized below to determine the probability that a subject randomly selected has a false positive result.

	Did the subject use marijuana?		Totals
	Yes	No	
Positive test result	119	24	143
Negative test result	3	154	157
Total	122	178	300

Example 32: Use the data summarized below to determine the probability that a randomly selected subject has managed to graduate from a most selective school.

Race	Graduated?	Nonselective	Selective	Most Selective	Total
White	Yes	1506	15014	5131	21651
	No	905	2343	281	3529
Black	Yes	181	1633	461	2275
	No	200	592	81	873
Hispanic	Yes	141	1983	429	2553
	No	208	465	45	718
Asian	Yes	17	199	323	539
	No	14	24	14	52
Total		3172	22253	6765	32190

Example 33: A bag contains 6 red marbles, 3 blue marbles, and 7 green marbles. If a marble is randomly selected from the bag, what is the probability that it is blue?

- A. $\frac{1}{3}$
- B. $\frac{1}{7}$
- C. $\frac{3}{16}$
- D. $\frac{1}{13}$

Solution: C. $\frac{3}{16}$

Example 34: A bag contains 3 red marbles, 4 blue marbles, and 6 green marbles. If a marble is randomly selected from the bag, what is the probability that it is **not** blue?

- A. **9/13**
- B. **9**
- C. **13/9**
- D. **4/13**



Solution: A. 9/13

**The box below gives an important recap of basic probability. Notice that basic probability is a single fraction and deals with a single event.

Basic Probability

-Key Words: Find the probability... “1” randomly selected...

-Formula(s):

$$P(A) = \frac{\text{\# of times "A" happened}}{\text{Total \# of observations}} \quad \text{OR} \quad P(A) = \frac{\text{\# of ways "A" can happen}}{\text{Total \# of possible outcomes}}$$

Unusualness:

We will define (arbitrarily) an unusual event as one that only occurs 5% or less of the time. Based on that idea answer the following question:

Example 35: Assume that one student in a class of 27 students is randomly selected to win a prize. Would it be “unusual” for you to win? (Assume “unusual” is a probability less than or equal to 0.05)

- A. **Yes**
- B. **No**

Solution: A. Yes

3.3 Fundamental Counting Rule and Combinations

Okay, let's take a little break from probability to look into a method of counting... I know you know how to count, but this is no ordinary counting—we are talking about Combinatorics here. In many probability problems, the big obstacle is finding the total number of outcomes, and this section presents several methods for finding such numbers without directly listing and counting the possibilities.

The Fundamental Counting Rule:

For a sequence of two events in which the first event can occur m ways and the second event can occur n ways, the events together can occur a total of $m \cdot n$ ways.

To use the fundamental counting rule, divide the task into a set of steps or actions, ask yourself how many options there are for each action, and multiply the resulting values.

Example 36: If you can choose three different shirts to go with a pair of pants and four different pairs of shoes, how many unique outfits can you create?

Solution: $3(4) = 12$ different outfits.



Before we introduce the next method, we need to cover some notation. The **factorial symbol, !** denotes the product of decreasing positive whole numbers.

Factorial:

$n! = n \times n-1 \times n-2 \times \dots \times 1$, for example $4! = 4 \times 3 \times 2 \times 1 = 24$

**note $0! = 1$ by definition.

The Factorial Counting Method:

A collection of n different items can be arranged in order $n!$ different ways. (This factorial rule reflects the fact that the first item may be selected from n different items, the second item may be selected from $n-1$ items, and so on.)

Example 37: In how many different orders can a UPS driver drop off five packages?



Example 37.5: In how many ways can I order six different numbers? For example, how many arrangements of {1, 3, 5, 7, 9, 11} are there?

The final counting rule we will discuss, is called the **Combinations Rule**.

Combinations Rule:

If a sample of r elements is to be drawn without replacement from a set of n elements, then the number of different samples possible is denoted by $nCr = \binom{n}{r} = \frac{n!}{r!(n-r)!}$

Three things are true of combinations:

- The combinations are formed by selecting r items from a set of n items.
- The order of the items in the subset does not matter.
- The selections are made without replacement, which means repetition of items is not allowed.

Example 38: If a sample of 5 elements is drawn from a set of 20 elements, *how many* different samples are there?

Solution: $\binom{20}{5} = \frac{20!}{5!(20-5)!} = \frac{20 \times 19 \times 18 \times \dots \times 3 \times 2 \times 1}{(5 \times 4 \times 3 \times 2 \times 1)(15 \times 14 \times 13 \times \dots \times 3 \times 2 \times 1)} = 15,504.$

Example 39: How many different sets of numbers can we form by choosing 6 numbers from the set of integers from 1 to 53? This is the same as the number of possible outcomes for a Florida Lotto drawing. What's the probability you would win with a randomly chosen six-digit string of numbers?



Example 39.5: If a bag contains 13 marbles numbered from 1 to 13, how many unique selections of three marbles are there?



Example 40: Find ${}_{10}C_2$

- A. **80,640**
- B. **40,320**
- C. **45**
- D. 5

Solution: C. 45

**Please note that the branch of mathematics called Combinatorics is quite vast, and we have only just scratched the surface.

The recap box below indicates that these questions will ask for 'the number of ways' something can be done. Look for that in problems you face on the test. The majority of the exam questions should say, 'find the probability', but not these counting questions.

Counting

-Key Words: "In how many ways ...", or "How many ..."

-Formula(s):

Fundamental Counting Rule:

of choices for 1st selection \times # choices for 2nd selection $\times \dots \times$ # choices for last selection

Factorial Rule: $n!$

Combinations: ${}_n C_r = \frac{n!}{r!(n-r)!}$ (order does not matter)

3.4 Additive Rule of Probability

Unions and Intersections

In this section we discuss some very basic ideas from set theory:

The **Union** of two events is the event that A or B or Both occur on a single performance of an experiment. We use the notation: $A \cup B$.

Think of a union as the merging of two sets. We simply create a larger set that contains all the elements of the two sets; however, we discard any repeats. That is, if both sets have the number 2 we only put one number 2 into the union.

Example 41: Find the union, $A \cup B$, given that: $A = \{1, 2, 3\}$ and $B = \{1, 4, 5\}$.

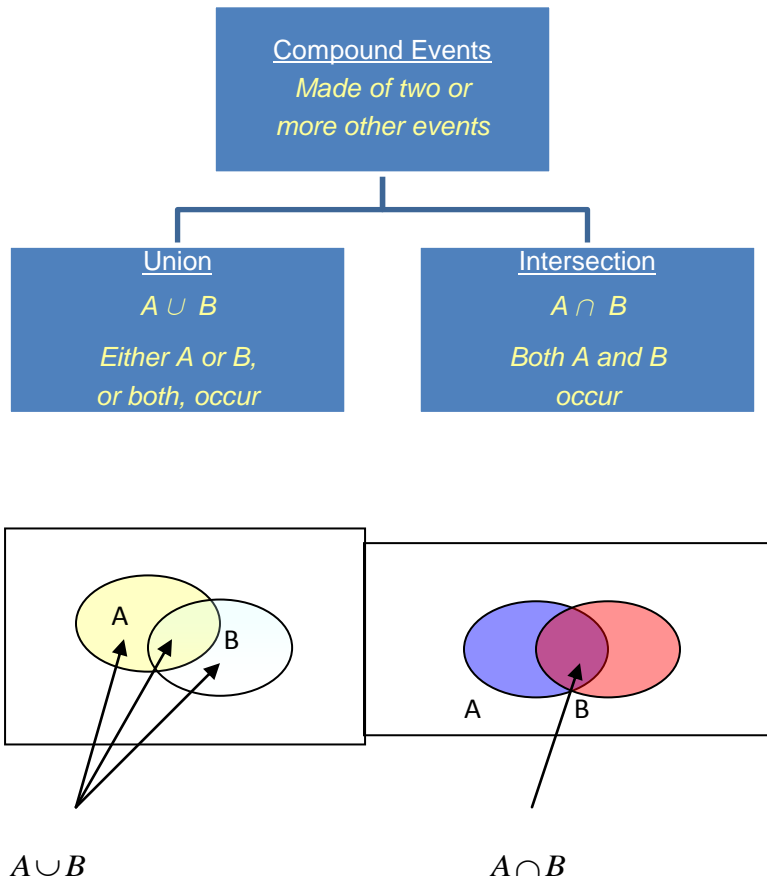
Solution: $A \cup B = \{1, 2, 3, 4, 5\}$

The **Intersection** of two events, A and B, is the event that occurs if both A and B occur on a single performance of the experiment. We use the notation: $A \cap B$

When thinking of an intersection, think of where the sets overlap or what the sets have in common.

Example 42: Find the intersection, $A \cap B$, given that: $A = \{1, 2, 3\}$ and $B = \{1, 4, 5\}$.

Solution: $A \cap B = \{1\}$

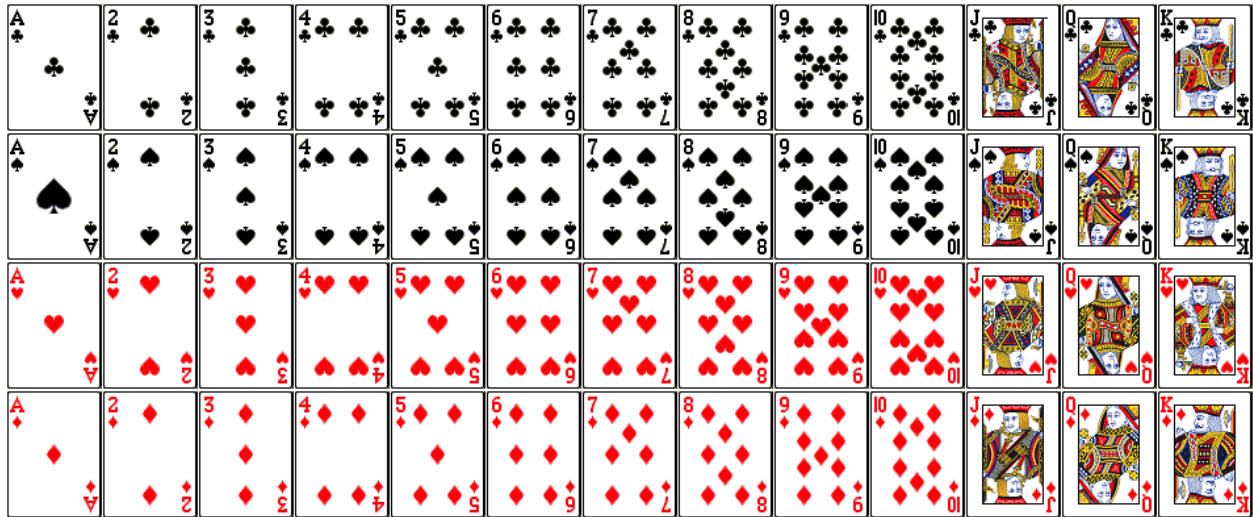


The Additive Rule and Mutually Exclusive Events

The main objective of this section is to present the addition rule as a device for finding probabilities that can be expressed as $P(A \text{ or } B)$, the probability that either event A occurs or event B occurs (or they both occur) as the single outcome of the procedure.

Addition Rule of Probability: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ where $P(A \cap B) = P(A \text{ and } B)$, which denotes the probability that A and B both occur at the same time as an outcome in a trial or procedure.

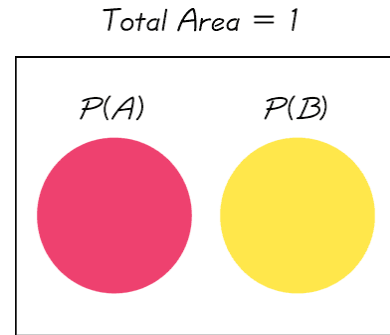
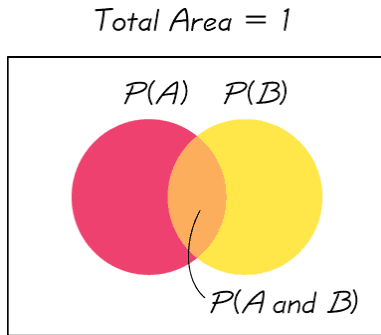
Example 43: Find the probability of randomly selecting one card from a deck of 52 cards and getting either a face card or a heart.



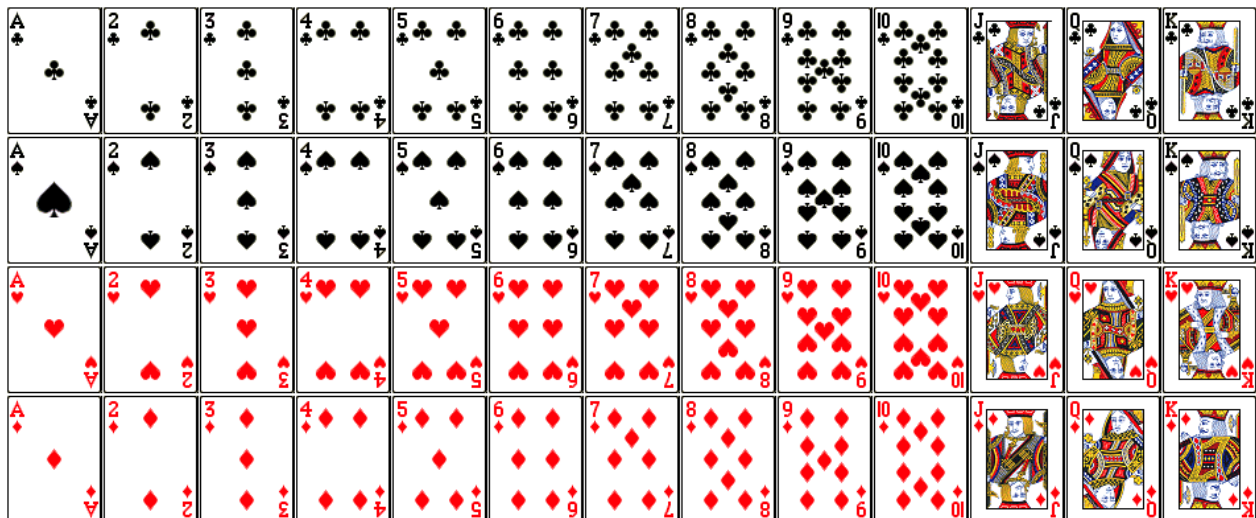
Two events are **mutually exclusive** if they cannot occur together at the same time.

If A and B are mutually exclusive events, $P(A \cap B) = 0$. When two events are mutually exclusive, we say **Events A and B are disjoint**. (That is, disjoint events do not overlap.)

The Venn Diagrams below show the difference between sets that overlap and those that do not:



Example 44: Find the probability of randomly selecting one card from a deck of 52 cards and getting either a face card or a three.



Example 45: (Use the table below which summarizes the results of a 2004 study) Find the probability of a single randomly selected subject being very happy or earning \$90,000 and over.

Study on Global Happiness					
Response	Under \$20,000	\$20,000-\$49,000	\$50,000-\$89,999	\$90,000 and over	Total
Not too Happy	55	64	20	6	145
Pretty Happy	192	280	129	55	656
Very Happy	70	149	108	45	372
Total	317	493	257	106	1173

Example 46: (Use the table below) Find the probability of a single randomly selected subject having a false positive or a false negative exam result.

Pregnancy Test Results			
	Positive result (Preg. Indicated)	Negative Result (Preg. Not Indicated)	Totals
Subject is pregnant	80	5	85
Subject is not Pregnant	3	11	14
Totals	83	16	99



Recap Box:

Addition Rule

-Key Words: Find the probability... “1” randomly selected... “or”...

-Formula: $P(A \text{ or } B) = P(A) + P(B) - P(A \cap B)$

$$P(A \text{ or } B) = \frac{\text{\# of subjects in group A}}{\text{Total}} + \frac{\text{\# of subjects in group B}}{\text{Total}} - \frac{\text{\# of subjects belonging to both A \& B}}{\text{Total}}$$

3.5 Conditional Probability

Additional information or other events occurring may have an impact on the probability of an event.



For example, if I ask you what is the chance I rolled a six on a die, you would say, “P(Rolling a 6) is one-sixth”, but if I told you an even number was rolled, the probability of a 6 goes up to one-third.

Conditional probability allows us to incorporate known information to more accurately reflect the likelihood a particular event occurs.

Conditional Probability Formula: To find the probability that event A occurs given event B occurs, we divide the probability that both event A and B occur by the probability that B occurs,

$$P(A | B) = \frac{P(A \cap B)}{P(B)}$$

Notation for Conditional Probability:

$P(B/A)$ represents the probability of event B occurring after it is assumed that event A has already occurred (read B/A as “ B given A .”)

Example 47: Find the probability of a subject having a false positive result given that the subject has a positive result.

	Did the subject use marijuana?		Totals
	Yes	No	
Positive test result	119	24	143
Negative test result	3	154	157
Total	122	178	300

Example 48: Find the probability of a subject getting a positive result given that the subject does smoke marijuana. Do the problem again given the person does not smoke marijuana.

	Did the subject use marijuana?		Totals
	Yes	No	
Positive test result	119	24	143
Negative test result	3	154	157
Total	122	178	300

Example 49: Find the probability of a person having a positive result on a pregnancy test given that they are not pregnant. Then find the probability of a person not being pregnant given that they have a positive pregnancy test. Are these probabilities the same?



Pregnancy Test Results			
	Positive result	Negative Result	Totals
	(Preg. Indicated)	(Preg. Not Indicated)	
Subject is pregnant	80	5	85
Subject is not Pregnant	3	11	14
Totals	83	16	99

Example 49.5: (Use the table below which summarizes the results of a 2004 study) Find the probability of a single randomly selected subject being very happy given that he/she earns \$90,000 and over per year. Then find the probability that a randomly selected individual is very happy given that he/she earns under \$20,000. What can we conclude?

Study on Global Happiness					
Response	Under \$20,000	\$20,000-\$49,000	\$50,000-\$89,999	\$90,000 and over	Total
Not too Happy	55	64	20	6	145
Pretty Happy	192	280	129	55	656
Very Happy	70	149	108	45	372
Total	317	493	257	106	1173

Example 50: The following table contains data from a study of two airlines which fly to Small Town, USA.

	Number of on time flights	Number of late flights
Podunk Airlines	33	6
Upstate Airlines	43	5

If one of the 87 flights is randomly selected, find the probability that the flight selected arrived on time given that it was an Upstate Airlines flight.

- A. $43/87$
- B. $11/76$
- C. $43/48$
- D. None of the above is correct.



Solution: C. $43/48$

Example 51: In a study, 55% of sampled executives had cheated at golf. The same study revealed that 20% of sampled executives had cheated at golf *and* lied in business. What is the probability that an executive had lied in business, *given* they had cheated in golf?



Independent Events

Two events A and B are independent if the occurrence of one does not affect the probability of the occurrence of the other. (Several events are similarly independent if the occurrence of any does not affect the probabilities of occurrence of the others.) If A and B are not independent, they are said to be **dependent**.

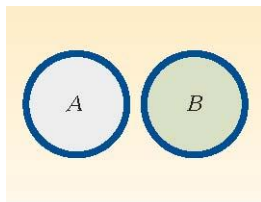
**Using the conditional probability formula to confirm independence: Two events are independent iff (if and only if),

$$P(A|B) = P(A) \text{ and } P(B|A) = P(B)$$

A very interesting and counter intuitive result is that two events that are mutually exclusive meaning they do not occur together are as a result dependent! This bothers most people, but the answer lies in the fact that mutually exclusive events have no intersection.

Recall: Events A and B are **mutually exclusive** if $A \cap B$ contains no sample points.

This leads to the idea that mutually exclusive events are dependent: $P(B|A) = 0$. If we know A has occurred there is no chance that B occurred since there is no intersection between the two.



5

Example: A common error in logic made when playing games of chance like dice is to think that the dice have a memory. Statements like, "it's been odd four rolls in a row, an even is due" assume that what happened during a previous roll affects what will happen on the next roll. Let's look at a simple example when rolling a single die twice. Find the probability of rolling an even given that an odd was rolled first. If you rolled ten odds in a row, would an even be "due?"



Here are the 36 possible outcomes for two rolls:

1, 1	2, 1	3, 1	4, 1	5, 1	6, 1
1, 2	2, 2	3, 2	4, 2	5, 2	6, 2
1, 3	2, 3	3, 3	4, 3	5, 3	6, 3
1, 4	2, 4	3, 4	4, 4	5, 4	6, 4
1, 5	2, 5	3, 5	4, 5	5, 5	6, 5
1, 6	2, 6	3, 6	4, 6	5, 6	6, 6

$$P(E|O) = \frac{P(E \cap O)}{P(O)} = \frac{\frac{9}{36}}{\frac{6}{6}} = \frac{9}{36} \cdot \frac{6}{3} = \frac{1}{2} = P(E)$$

This result tells us that the two rolls of the die are independent events (actually, we also need to show the same holds for $P(O|E)$).

Recap Box:

Conditional Probability

-Key Words: Find the probability... “1” randomly selected... “given that”...

-Formula: $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\text{Number of Subjects in Both Groups/Total}}{\text{Number of Subjects in Group B/Total}}$

*short cut: Only look at the numbers in the row or column relating to the “given that” condition, the total of that row or column is your denominator, then just get your numerator.

3.6 Multiplicative Rule of Probability

The Multiplicative Rule and Independent Events

The conditional probability formula can be rearranged into the Multiplicative Rule of Probability to find joint probability.

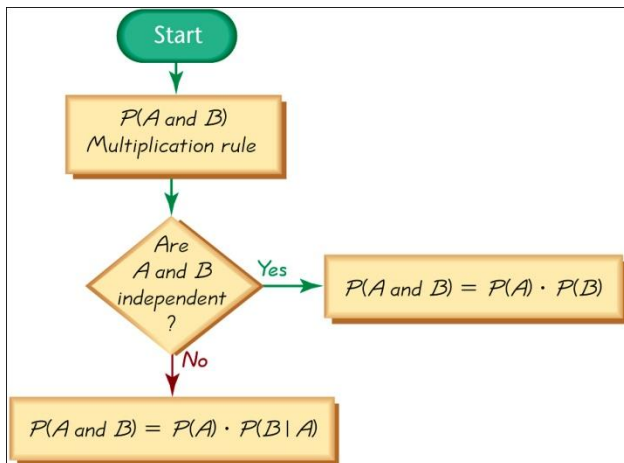
Multiplication Rule for Probability: The probability that two events A and B occur is given by:

$$P(A \text{ and } B) = P(A \cap B) = P(A) \bullet P(B | A)$$

If the two events A and B are **independent**, we may use:

$$P(A \text{ and } B) = P(A \cap B) = P(A) \bullet P(B)$$

If the outcome of the first event A somehow affects the probability of the second event B, it is important to adjust the probability of B to reflect the occurrence of event A (*only use the independent formula for events that are independent*).



Example 52: Find the probability of guessing the correct answers on three different multiple choice questions that have 5 answer choices each.

Example 52.5: In a recent Pew Research Center study, only 11% of 18 – 24 year olds reported that their current job is a career. For participants in the 30 – 34 age group the percentage was 49%. Use the results to estimate the probability of four randomly selected 30 – 34 year olds all feeling that their current job is a career.

Example 53: If a store has 20 DVD players on a shelf that work perfectly and 3 that are defective, find the probability of randomly selecting two defective DVD players when purchasing a pair of DVD players from the store.



Example 53.5: An explosive disposal unit has to find two bombs which are hidden in two cars parked in a lot with five other cars. If the bomb squad only has time to search three cars before the area must be evacuated, what is the probability that none of the three cars randomly searched contains a bomb?



If a sample size is no more than 5% of the size of the population, treat the selections as being independent (even if the selections are made without replacement, so they are technically dependent).

Recap Box:

Multiplication Rule

-**Key Words:** Find the probability... (more than 1) randomly selected

- For example; 4 randomly selected ____ · ____ · ____ · ____

-**Formula:** $P(A \cap B) = P(A) \cdot P(B)$; (Independent, **with** replacement)

OR

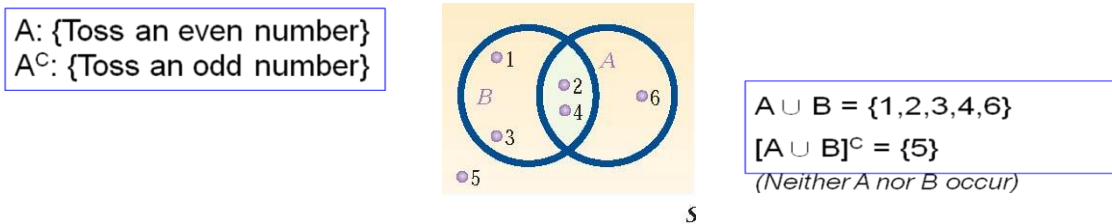
$P(A \cap B) = P(A) \cdot P(B / A)$; (Dependent, **without** replacement)

3.7 Probability of At Least One

Complementary Events

The **Complement** of an event A is the event that A does not occur—All the sample points in the sample space not in event A.

Notation: $A^c = \bar{A} = A' = A \text{ complement}$



Rule of Complements: $P(A) + P(\bar{A}) = 1$

Using a small bit of Algebra, we have:
 $P(A) = 1 - P(\bar{A})$
 $P(\bar{A}) = 1 - P(A)$

Example 54: If the probability of losing money on a business venture is 20%, the probability of making money is 35%, and the probability of breaking even is 45%, what is the probability of not losing money?



An important rule of probability is the **probability of “at least one”**. The rule is as follows:

$$P(\text{at least one}) = 1 - P(\text{none})$$

Example 55: What is the probability when flipping a coin twice that at least one head turns up?



Solution: Define event A as A: {At least one head on two coin flips} and event A complement as A^C : {No heads}, then $P(A) = 1 - P(A^C)$. Now create a list of all of the equally likely outcomes for the experiment of flipping a coin twice: HH, HT, TH, and TT. Since there are 4 outcomes, and only TT that belongs to A complement, we can say $P(A) = 1 - P(A^C) = 1 - \frac{1}{4} = \frac{3}{4}$.

*Another way to solve the problem would be to use the direct approach: since $A : \{HH, HT, TH\}$, $P(A) = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4}$. Please understand in these simple problems it is feasible to work the problem with either approach; however, that is not always the case. It is often necessary to apply the **probability of “at least one”** rule given above.

Example 56: Find the probability of awaking in the morning when using three alarm clocks if each alarm clock has a 99% chance of waking you.



Example 57: In order for a container of Sony HDTV sets to be offloaded from a truck for sale at a retail store, a random selection of three sets is inspected from the container. If the three sets are defect free, the entire container will be accepted without further inspection. If an individual Sony HDTV has a 0.2% defective rate, what is the probability that a container of Sony HDTV sets will require further inspection?



Recap box:

Probability of At Least One

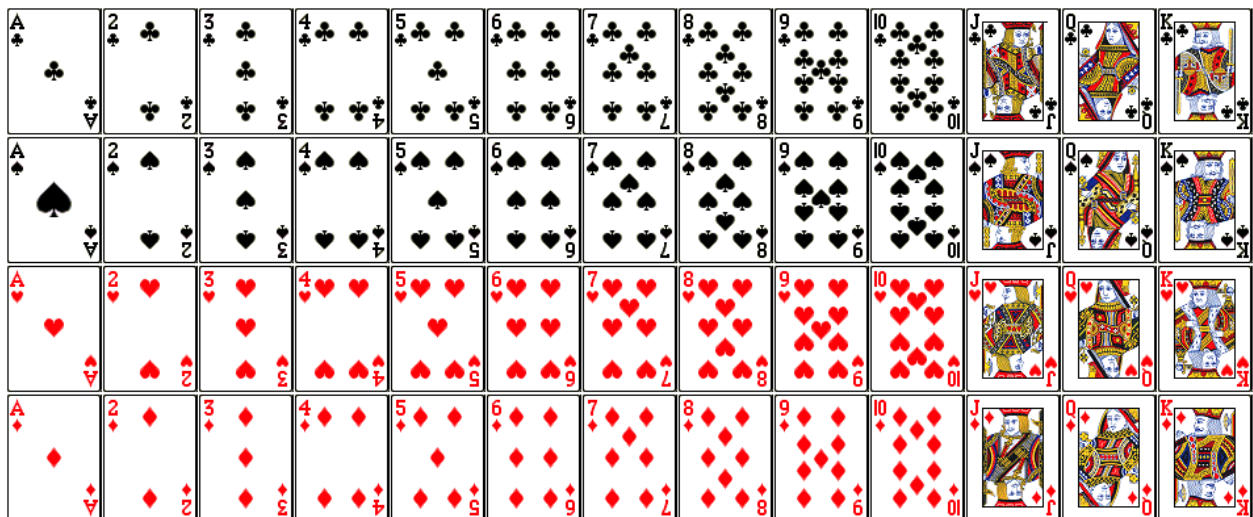
-**Key Words:** Sometimes it will say, “Find the probability that at least one...”

-**Formula:** $P(\text{At Least } 1) = 1 - P(\text{None})$

3.8 Advanced Problems

Example 58: After shuffling a deck of cards well, does selecting five cards from the deck constitute a random sample of five cards? If so, what is the probability that any particular sample of five cards is chosen?

Example 58.5: What is the probability you draw a four-of-a-kind when choosing five cards from a deck of 52 cards (for example: 4 kings and the 3 of hearts is a four-of-a-kind hand)?



Example 59: If a box contains 8 identical red poker chips and 5 identical blue poker chips, what is the probability that two randomly selected poker chips taken from the box without replacement are the same color?