Section 6.1

Recall the fundamental identities:

Fundamental Identities

The **reciprocal identities**:

$$\sin\theta = \frac{1}{\csc\theta} \qquad \cos\theta = \frac{1}{\sec\theta} \qquad \tan\theta = \frac{1}{\cot\theta}$$

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

The **quotient identities**:

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

Even-Odd Identities

The cosine and secant functions are even.

$$\cos(-t) = \cos t \qquad \sec(-t) = \sec t$$

The sine, cosecant, tangent, and cotangent functions are odd.

$$\sin(-t) = -\sin t \qquad \csc(-t) = -\csc t$$

$$\tan(-t) = -\tan t \qquad \cot(-t) = -\cot t$$

The Pythagorean Identities

$$\sin^2\theta + \cos^2\theta = 1 \qquad 1 + \tan^2\theta = \sec^2\theta \qquad 1 + \cot^2\theta = \csc^2\theta$$

Using Fundamental Identities to Verify Other Identities

The fundamental trig identities are used to establish other relationships among trigonometric functions. To verify an identity we show that one side of the identity can be simplified so that is identical to the other side. Each side is manipulated independently of the other side of the equation. Usually it is best to start with the more complicated side of the identity.

Example 42 **Changing to Sines and Cosines to Verify an Identity**

Verify the identity: sec x cot x = csc x.

Solution The left side of the equation contains the more complicated expression. Thus, we work with the left side. Let us express this side of the identity in terms of sines and cosines. Perhaps this strategy will enable us to transform the left side into csc x, the expression on the right.

$$\sec x \cot x = \frac{1}{\cos x} \bullet \frac{\cos x}{\sin x}$$

$$= \frac{1}{\sin x} = \csc x$$

Example 43

Verify the identity: $\sin x \tan x + \cos x = \sec x$

Example 44 **Using Factoring to Verify an Identity**

Verify the identity: $\cos x - \cos x \sin^2 x = \cos^3 x$

Solution We start with the more complicated side, the left side. Factor out the greatest common factor, $\cos x$, from each of the two terms.

$\cos x - \cos x \sin^2 x = \cos x(1 - \sin^2 x)$ **Factor cos x from the two terms.**

$= \cos x \cdot \cos^2 x$ **Use a variation of $\sin^2 x + \cos^2 x = 1$. Solving for $\cos^2 x$, we obtain $\cos^2 x = 1 - \sin^2 x$.**

$= \cos^3 x$ **Multiply.**

Example 45 **Combining Fractional Expressions to Verify an Identity**

Verify the identity: $\dfrac{\cos x}{1+\sin x}+\dfrac{1+\sin x}{\cos x}=2\sec x$

Example 46 **Multiplying the Numerator and Denominator by the Same Factor to Verify an Identity** (think rationalizing the numerator or denominator)

Verify the identity: $\dfrac{\sin x}{1+\cos x}=\dfrac{1-\cos x}{\sin x}$

Example 47 **Changing to Sines and Cosines to Verify an Identity**

Verify the identity: $\dfrac{\tan x-\sin(-x)}{1+\cos x}=\tan x$

Example 48 **Working with Both Sides Separately to Verify an Identity**

Verify the identity: $\dfrac{1}{1+\cos\theta}+\dfrac{1}{1-\cos\theta}=2+2\cot^2\theta$

Guidelines for Verifying Trigonometric Identities

1. Work with each side of the equation independently of the other side. Start with the more complicated side and transform it in a step-by-step fashion until it looks exactly like the other side.
2. Analyze the identity and look for opportunities to apply the fundamental identities. Rewriting the more complicated side of the equation in terms of sines and cosines is often helpful.
3. If sums or differences of fractions appear on one side, use the least common denominator and combine the fractions.
4. Don't be afraid to stop and start over again if you are not getting anywhere.

    Creative puzzle solvers know that strategies leading to dead ends often provide good problem-solving ideas.

Section 6.2 **Sum and Difference Formulas**

The Cosine of the Difference of Two Angles

$$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$$

The cosine of the difference of two angles equals the cosine of the first angle times the cosine of the second angle plus the sine of the first angle times the sine of the second angle.

Example 49

Use the difference formula for Cosines to find the Exact Value:

Find the exact value of cos 15°

Solution We know exact values for trigonometric functions of 60° and 45°. Thus, we write 15° as 60° - 45° and use the difference formula for cosines.

cos 15° = cos(60° - 45°) = cos 60° cos 45° + sin 60° sin 45°

Example 50

Find the exact value of cos 80° cos 20° + sin 80° sin 20°.

Example 51

Find the exact value of cos(180º-30º)

Example 52

Verify the following identity: $\dfrac{\cos(\alpha-\beta)}{\sin\alpha\cos\beta}=\cot\alpha+\tan\beta$

Example 53

Verify the following identity: $\cos\left(x-\dfrac{5\pi}{4}\right)=-\dfrac{\sqrt{2}}{2}(\cos x+\sin x)$

Sum and Difference Formulas for Cosines and Sines

$$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$$
$$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$$
$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$$
$$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$$

Example 54

Find the exact value of sin(30º+45º)

Example 55

Find the exact value of $\sin\frac{7\pi}{12}$

Example 56

Show that $\sin\left(x-\frac{3\pi}{2}\right)=\cos x$

Sum and Difference Formulas for Tangents

The tangent of the sum of two angles equals the tangent of the first angle plus the tangent of the second angle divided by 1 minus their product.

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

$$\tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$$

The tangent of the difference of two angles equals the tangent of the first angle minus the tangent of the second angle divided by 1 plus their product.

Example 57

Find the exact value of tan(105º)

Example 58

Verify the identity: $\tan\left(x - \frac{\pi}{4}\right) = \frac{\tan x - 1}{\tan x + 1}$

Example 59

Write the following expression as the sine, cosine, or tangent of an angle. Then find the exact value of the expression.

$$\sin\frac{7\pi}{12}\cos\frac{\pi}{12} - \cos\frac{7\pi}{12}\sin\frac{\pi}{12}$$

Section 6.3 Double-Angle and Half-Angle Formulas

Double – Angle Formulas

$$\sin 2\theta = 2\sin\theta\cos\theta$$
$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$
$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$$

We can derive these by using the sum formulas we learned in section 6.2.

Example 60

If $\sin\theta = \frac{5}{13}$ and θ lies in quadrant II, find the exact value of:

a. $\sin 2\theta$ b. $\cos 2\theta$ c. $\tan 2\theta$

Example 61

Find the exact value of $\frac{2\tan 15°}{1-\tan^2 15°}$

Three Forms of the Double-Angle Formula for cos2θ

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$
$$\cos 2\theta = 2\cos^2\theta - 1$$
$$\cos 2\theta = 1 - 2\sin^2\theta$$

Example 62

Verify the identity: $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

Power-Reducing Formulas

$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$
$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$
$$\tan^2\theta = \frac{1-\cos 2\theta}{1+\cos 2\theta}$$

Example 63

Write an expression for $\cos^4\theta$ that does not have powers on the trigonometric functions greater than 1.

Example 64

Write an equivalent expression for $\sin^4 x$ that does not contain powers of trigonometric functions greater than 1.

$$\sin^4 x = \sin^2 x \sin^2 x = \left(\frac{1-\cos 2x}{2}\right)\left(\frac{1-\cos 2x}{2}\right)$$

$$\left(\frac{1-2\cos 2x+\cos^2 2x}{4}\right) = \left(\frac{1-2\cos 2x+\dfrac{1+\cos 2x}{2}}{4}\right)$$

$$= \left(\frac{2-4\cos 2x+1+\cos 2x}{8}\right) = \frac{(3-3\cos 2x)}{8}$$

Half-Angle Identities

$$\sin\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{2}}$$

$$\cos\frac{x}{2} = \pm\sqrt{\frac{1+\cos x}{2}}$$

$$\tan\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{1+\cos x}} = \frac{\sin x}{1+\cos x} = \frac{1-\cos x}{\sin x}$$

where the sign is determined by the quadrant in which $\frac{x}{2}$ lies.

Example 65

Find the exact value of $\cos 112.5°$

Solution Because 112.5° = $^{225°}$**/2**, we use the half-angle formula for cos $^{\alpha}$**/2** with α = 225°. What sign should we use when we apply the formula? Because 112.5° lies in quadrant II, where only the sine and cosecant are positive, cos 112.5° < 0. Thus, we use the - sign in the half-angle formula.

Example 66

Verify the identity: $\tan\theta = \dfrac{1-\cos 2\theta}{\sin 2\theta}$

Half-Angle Formulas for:

$$\tan\frac{\alpha}{2} = \frac{1-\cos\alpha}{\sin\alpha}$$
$$\tan\frac{\alpha}{2} = \frac{\sin\alpha}{1+\cos\alpha}$$

Example 67

Verify the identity: $\tan\dfrac{\alpha}{2} = \csc\alpha - \cot\alpha$

Example 68

Verify the following identity: $(\sin\theta - \cos\theta)^2 = 1 - \sin 2\theta$

Solution:

$$\begin{aligned}
&(\sin\theta-\cos\theta)^2 \\
&= \sin^2\theta - 2\sin\theta\cos\theta + \cos^2\theta \\
&= \frac{1-\cos 2\theta}{2} + \frac{1+\cos 2\theta}{2} - 2\sin\theta\cos\theta \\
&= \frac{2}{2} - 2\sin\theta\cos\theta = 1 - \sin 2\theta
\end{aligned}$$

Section 6.4 **Product-to-Sum and Sum-to-Product Formulas**

Product-to-Sum Formulas

$$\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta)-\cos(\alpha+\beta)]$$

$$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha-\beta)+\cos(\alpha+\beta)]$$

$$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha-\beta)]$$

$$\cos\alpha\sin\beta = \frac{1}{2}[\sin(\alpha+\beta)-\sin(\alpha-\beta)]$$

Example 69

Express each of the following as a sum or a difference:

a. $\sin 8x \sin 3x$ b. $\sin 4x \cos x$

Example 70

Express the following product as a sum or difference:

$$\cos 3x \cos 2x$$

Sum-to-Product Formulas

$$\sin\alpha+\sin\beta=2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$

$$\sin\alpha-\sin\beta=2\sin\frac{\alpha-\beta}{2}\cos\frac{\alpha+\beta}{2}$$

$$\cos\alpha+\cos\beta=2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$

$$\cos\alpha-\cos\beta=-2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$$

Example 71

Express each sum or difference as a product:

a. $\sin 9x+\sin 5x$ b. $\cos 4x-\cos 3x$

Example 72

Express the difference as a product:

$\sin 4x-\sin 2x$

Example 73

Verify the following identity:

$$\frac{\sin x+\sin y}{\sin x-\sin y}=\tan\frac{x+y}{2}\cot\frac{x-y}{2}$$

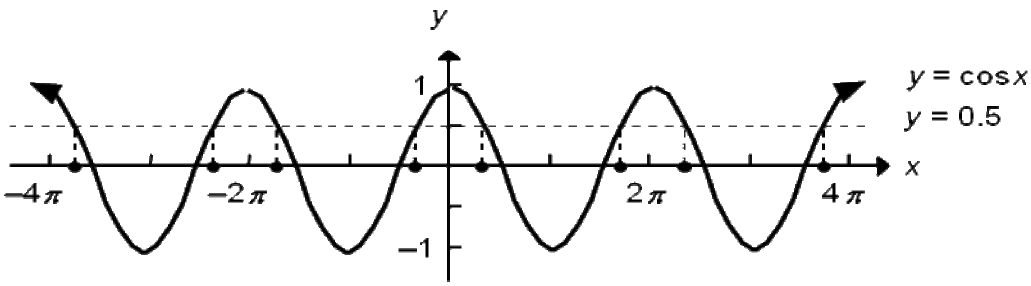
Example 74

Verify the following identity:

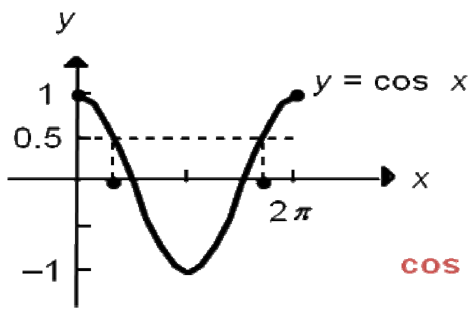
$$\frac{\cos 3x-\cos 5x}{\sin 3x+\sin 5x}=\tan x$$

Section 6.5 **Trigonometric Equations**

This section involves equations that have a trigonometric expression with a variable, such as $\cos x$. To understand this section we must consider a simple equation such as $\cos x = 0.5$.



cos x = 0.5 has infinitely many solutions for $-\infty < x < \infty$



cos x = 0.5 has two solutions for $0 < x < 2\pi$

Equations Involving a Single Trigonometric Function

To solve an equation containing a single trigonometric function:

- Isolate the function on one side of the equation.
- Solve for the variable.

Example 75

Solve the equation: $3 \sin x - 2 = 5 \sin x - 1$.

Solution The equation contains a single trigonometric function, $\sin x$.

Step 1 **Isolate the function on one side of the equation.** We can solve for $\sin x$ by collecting all terms with $\sin x$ on the left side, and all the constant terms on the right side.

$3 \sin x - 2 = 5 \sin x - 1$ This is the given equation.

$3 \sin x - 5 \sin x - 2 = 5 \sin x - 5 \sin x - 1$ Subtract 5 sin x from both sides.

$-2 \sin x - 2 = -1$ Simplify.

$-2 \sin x = 1$ Add 2 to both sides.

$\sin x = -1/2$ Divide both sides by -2 and solve for sin x.

Example 76 Equations with a multiple angle

Solve the equation: $\tan 3x = 1$, within the interval: $0 \le x < 2\pi$

Example 77 Equations with a multiple angle

$$\sin \frac{x}{2} = \frac{\sqrt{3}}{2}, \quad 0 \le x < 2\pi$$

Example 78

Solve the equation: $2\cos^2 x + \cos x - 1 = 0, \quad 0 \le x < 2\pi.$

Example 79

$\tan x \sin^2 x = 3\tan x$, within $0 \le x < 2\pi$

Example 80 Using a trigonometric identity to solve a trig equation:

$\cos 2x + 3\sin x - 2 = 0$, within $0 \le x < 2\pi$

Example 81 Using a trigonometric identity to solve a trig equation:

$\sin x \cos x = \frac{1}{2}$, within $0 \le x < 2\pi$

Example 82 Using a trigonometric identity to solve a trig equation:

$\sin x - \cos x = 1$, within $0 \le x < 2\pi$

Example 83 Solve the following equation:

$7\cos\theta + 9 = -2\cos\theta$

Example 84 Solve the equation on the interval [0,2π):

$$\tan\frac{\theta}{2} = \frac{\sqrt{3}}{3}$$

Example 85 Solve the equation on the interval [0,2π)

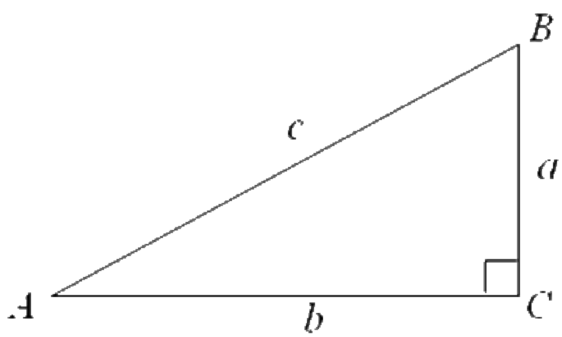
$$\sin 2x = \sin x$$

Example 86   Solve the equation on the interval $[0,2\pi)$

$$\cos^2 x + 2\cos x - 3 = 0$$

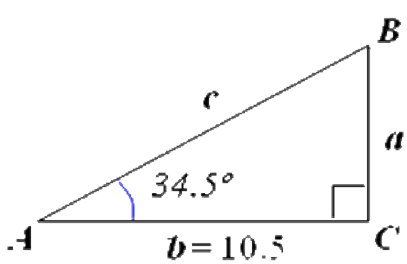
Section 5.8

Solving a right triangle means finding the missing lengths of the sides and the measurements of its angles. We will label the right triangle as is done in the following diagram:



Example 87

Solve the right triangle shown.



Solution We begin by finding the the measure of angle B. We do not need a trigonometric function to do so. Because $C = 90°$ and the sum of a triangle's angles is 180, we see that $A + B = 90°$. Thus,

$$B = 90° - A = 90° - 34.5° = 55.5°.$$

Now we need to find a. Because we have a known angle, and unknown opposite side, and a known adjacent side, we use the tangent function.

$$\tan 34.5° = a/10.5$$

Now we solve for a.

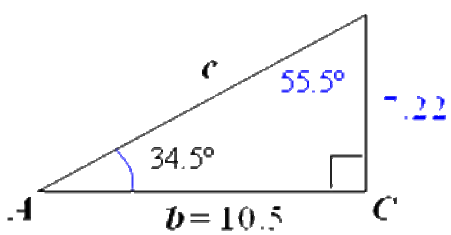
A= 10.5tan34.5°=7.22

Finally, we need to find c. Because we have a known angle, a known adjacent side, and an unknown hypotenuse, we use the cosine function.

cos34.5° = 10.5/c

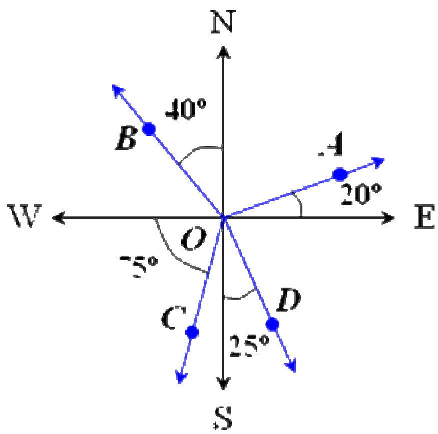
c=10.5/cos34.5° = 12.74

In summary, B = 55.5°, a = 7.22, and c = 12.74.



Example 88

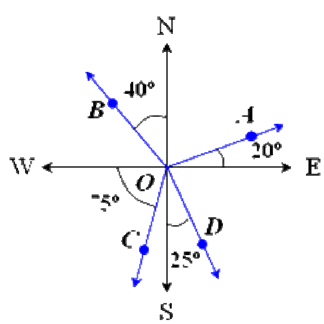
Use the figure to find: **a**. the bearing from O to B. **b**. the bearing from O to A.



Solution

a. To find the bearing from O to B, we need the acute angle between the ray OB and the north-south line through O. The measurement of this angle is given to be 40°. The figure shows that the angle is measured from the north side of the north-south line and lies west of the north-south line. Thus, the bearing from O to B is N 40° W.

Use the figure to find: **a**. the bearing from O to B. **b**. the bearing from O to A.



Solution

b. To find the bearing from O to A, we need the acute angle between the ray OA and the north-south line through O. This angle is specified by the voice balloon in the figure. The figure shows that this angle measures 90° – 20°, or 70°. This angle is measured from the north side of the north-south line. This angle is also east of the north-south line. This angle is also east of the north-south line. Thus the bearing from O to A is N 70° E.

Example 89

A 200 foot cliff drops vertically into the ocean. If the angle of elevation of a ship to the top of the cliff is 22.3 degrees, how far above shore is the ship?

Example 90

A building that is 250 feet high cast a shadow that is 40 feet long. Find the angle of elevation of the sun at that time.

Example 91

A boat leaves the entrance to a harbor and travels 40 miles on a bearing $S\,64°\,E$. How many miles south and how many miles east from the harbor has the boat traveled?