List of integrals of trigonometric functions

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The following is a list of integrals (antiderivative functions) of trigonometric functions. For antiderivatives involving both exponential and trigonometric functions, see List of integrals of exponential functions. For a complete list of antiderivative functions, see lists of integrals. See also trigonometric integral.

Generally, if the function $\sin(x)$ is any trigonometric function, and $\cos(x)$ is its derivative,

$$\int a \cos nx \, dx = \frac{a}{n} \sin nx + C$$

In all formulas the constant a is assumed to be nonzero, and C denotes the constant of integration.

Contents

Integrals involving only sine

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax + C$$

$$\int \sin^2 ax \, dx = \frac{x}{2} - \frac{1}{4a} \sin 2ax + C = \frac{x}{2} - \frac{1}{2a} \sin ax \cos ax + C$$

$$\int \sin^3 ax \, dx = \frac{\cos 3ax}{12a} - \frac{3 \cos ax}{4a} + C$$

$$\int x \sin^2 ax \, dx = \frac{x^2}{4} - \frac{x}{4a} \sin 2ax - \frac{1}{8a^2} \cos 2ax + C$$

$$\int x^2 \sin^2 ax \, dx = \frac{x^3}{6} - \left(\frac{x^2}{4a} - \frac{1}{8a^3} \right) \sin 2ax - \frac{x}{4a^2} \cos 2ax + C$$

$$\int \sin b_1 x \sin b_2 x \, dx = \frac{\sin((b_1 - b_2)x)}{2(b_1 - b_2)} - \frac{\sin((b_1 + b_2)x)}{2(b_1 + b_2)} + C \qquad \text{(for } |b_1| \neq |b_2|\text{)}$$

$$\int \sin^n ax \, dx = -\frac{\sin^{n-1} ax \cos ax}{na} + \frac{n-1}{n} \int \sin^{n-2} ax \, dx \qquad \text{(for } n > 2\text{)}$$

$$\int \frac{dx}{\sin ax} = \frac{1}{a} \ln \left| \tan \frac{ax}{2} \right| + C$$

$$\int \frac{\mathrm{d}x}{\sin^n ax} = \frac{\cos ax}{a(1-n)\sin^{n-1} ax} + \frac{n-2}{n-1}\int \frac{\mathrm{d}x}{\sin^{n-2} ax} \qquad \text{(for } n > 1\text{)}$$

$$\int x\sin ax\,\mathrm{d}x = \frac{\sin ax}{a^2} - \frac{x\cos ax}{a} + C$$

$$\int x^n \sin ax\,\mathrm{d}x = -\frac{x^n}{a}\cos ax + \frac{n}{a}\int x^{n-1}\cos ax\,\mathrm{d}x = \sum_{k=0}^{2k\leq n}(-1)^{k+1}\frac{x^{n-2k}}{a^{1+2k}}\frac{n!}{(n-2k)!}\cos ax + \sum_{k=0}^{2k+1\leq n}(-1)^k\frac{x^{n-1-2k}}{a^{2+2k}}\frac{n!}{(n-2k-1)!}\sin a$$

$$\int_{-\frac{a}{2}}^{\frac{a}{2}} x^2\sin^2\frac{n\pi x}{a}\,\mathrm{d}x = \frac{a^3\left(n^2\pi^2 - 6\right)}{24n^2\pi^2} \qquad \text{(for } n = 2, 4, 6...\text{)}$$

$$\int \frac{\sin ax}{x}\mathrm{d}x = \sum_{n=0}^{\infty}(-1)^n\frac{(ax)^{2n+1}}{(2n+1)\cdot(2n+1)!} + C$$

$$\int \frac{\sin ax}{x^n}\mathrm{d}x = -\frac{\sin ax}{(n-1)x^{n-1}} + \frac{a}{n-1}\int \frac{\cos ax}{x^{n-1}}\mathrm{d}x$$

$$\int \frac{\mathrm{d}x}{1\pm\sin ax} = \frac{1}{a}\tan\left(\frac{ax}{2}\mp\frac{\pi}{4}\right) + C$$

$$\int \frac{x\,\mathrm{d}x}{1+\sin ax} = \frac{x}{a}\tan\left(\frac{ax}{2}-\frac{\pi}{4}\right) + \frac{2}{a^2}\ln\left|\cos\left(\frac{ax}{2}-\frac{\pi}{4}\right)\right| + C$$

$$\int \frac{x\,\mathrm{d}x}{1-\sin ax} = \frac{x}{a}\cot\left(\frac{\pi}{4}-\frac{ax}{2}\right) + \frac{2}{a^2}\ln\left|\sin\left(\frac{\pi}{4}-\frac{ax}{2}\right)\right| + C$$

$$\int \frac{\sin ax\,\mathrm{d}x}{1\pm\sin ax} = \pm x + \frac{1}{a}\tan\left(\frac{\pi}{4}\mp\frac{ax}{2}\right) + C$$

Integrands involving only cosine

$$\int \cos ax\,\mathrm{d}x = \frac{1}{a}\sin ax + C$$

$$\int \cos^2 ax\,\mathrm{d}x = \frac{x}{2} + \frac{1}{4a}\sin 2ax + C = \frac{x}{2} + \frac{1}{2a}\sin ax\cos ax + C$$

$$\int \cos^n ax\,\mathrm{d}x = \frac{\cos^{n-1} ax\sin ax}{na} + \frac{n-1}{n}\int \cos^{n-2} ax\,\mathrm{d}x \qquad \text{(for } n > 0\text{)}$$

$$\int x\cos ax\,\mathrm{d}x = \frac{\cos ax}{a^2} + \frac{x\sin ax}{a} + C$$

$$\int x^2\cos^2 ax\,\mathrm{d}x = \frac{x^3}{6} + \left(\frac{x^2}{4a} - \frac{1}{8a^3}\right)\sin 2ax + \frac{x}{4a^2}\cos 2ax + C$$

$$\int x^n\cos ax\,\mathrm{d}x = \frac{x^n\sin ax}{a} - \frac{n}{a}\int x^{n-1}\sin ax\,\mathrm{d}x = \sum_{k=0}^{2k+1\leq n}(-1)^k\frac{x^{n-2k-1}}{a^{2+2k}}\frac{n!}{(n-2k-1)!}\cos ax + \sum_{k=0}^{2k\leq n}(-1)^k\frac{x^{n-2k}}{a^{1+2k}}\frac{n!}{(n-2k)!}\sin ax$$

$$\int \frac{\cos ax}{x}\mathrm{d}x = \ln|ax| + \sum_{k=1}^{\infty}(-1)^k\frac{(ax)^{2k}}{2k\cdot(2k)!} + C$$

$$\int \frac{\cos ax}{x^n} \mathrm{d}x = -\frac{\cos ax}{(n-1)x^{n-1}} - \frac{a}{n-1} \int \frac{\sin ax}{x^{n-1}} \mathrm{d}x \qquad \text{(for } n \neq 1\text{)}$$

$$\int \frac{\mathrm{d}x}{\cos ax} = \frac{1}{a} \ln \left| \tan \left(\frac{ax}{2} + \frac{\pi}{4} \right) \right| + C$$

$$\int \frac{\mathrm{d}x}{\cos^n ax} = \frac{\sin ax}{a(n-1)\cos^{n-1} ax} + \frac{n-2}{n-1} \int \frac{\mathrm{d}x}{\cos^{n-2} ax} \qquad \text{(for } n > 1\text{)}$$

$$\int \frac{\mathrm{d}x}{1 + \cos ax} = \frac{1}{a} \tan \frac{ax}{2} + C$$

$$\int \frac{\mathrm{d}x}{1 - \cos ax} = -\frac{1}{a} \cot \frac{ax}{2} + C$$

$$\int \frac{x \, \mathrm{d}x}{1 + \cos ax} = \frac{x}{a} \tan \frac{ax}{2} + \frac{2}{a^2} \ln \left| \cos \frac{ax}{2} \right| + C$$

$$\int \frac{x \, \mathrm{d}x}{1 - \cos ax} = -\frac{x}{a} \cot \frac{ax}{2} + \frac{2}{a^2} \ln \left| \sin \frac{ax}{2} \right| + C$$

$$\int \frac{\cos ax \, \mathrm{d}x}{1 + \cos ax} = x - \frac{1}{a} \tan \frac{ax}{2} + C$$

$$\int \frac{\cos ax \, \mathrm{d}x}{1 - \cos ax} = -x - \frac{1}{a} \cot \frac{ax}{2} + C$$

$$\int \cos a_1 x \cos a_2 x \, \mathrm{d}x = \frac{\sin(a_1 - a_2)x}{2(a_1 - a_2)} + \frac{\sin(a_1 + a_2)x}{2(a_1 + a_2)} + C \qquad \text{(for } |a_1| \neq |a_2|\text{)}$$

Integrands involving only tangent

$$\int \tan ax \, \mathrm{d}x = -\frac{1}{a} \ln |\cos ax| + C = \frac{1}{a} \ln |\sec ax| + C$$

$$\int \tan^n ax \, \mathrm{d}x = \frac{1}{a(n-1)} \tan^{n-1} ax - \int \tan^{n-2} ax \, \mathrm{d}x \qquad \text{(for } n \neq 1\text{)}$$

$$\int \frac{\mathrm{d}x}{q \tan ax + p} = \frac{1}{p^2 + q^2} \left(px + \frac{q}{a} \ln |q \sin ax + p \cos ax| \right) + C \qquad \text{(for } p^2 + q^2 \neq 0\text{)}$$

$$\int \frac{\mathrm{d}x}{\tan ax + 1} = \frac{x}{2} + \frac{1}{2a} \ln |\sin ax + \cos ax| + C$$

$$\int \frac{\mathrm{d}x}{\tan ax - 1} = -\frac{x}{2} + \frac{1}{2a} \ln |\sin ax - \cos ax| + C$$

$$\int \frac{\tan ax \, \mathrm{d}x}{\tan ax + 1} = \frac{x}{2} - \frac{1}{2a} \ln |\sin ax + \cos ax| + C$$

$$\int \frac{\tan ax \, \mathrm{d}x}{\tan ax - 1} = \frac{x}{2} + \frac{1}{2a} \ln |\sin ax - \cos ax| + C$$

Integrands involving only secant

See Integral of the secant function.

$$\int \sec ax \, \mathrm{d}x = \frac{1}{a} \ln |\sec ax + \tan ax| + C$$

$$\int \sec^2 x \, \mathrm{d}x = \tan x + C$$

$$\int \sec^n ax \, \mathrm{d}x = \frac{\sec^{n-2} ax \tan ax}{a(n-1)} + \frac{n-2}{n-1} \int \sec^{n-2} ax \, \mathrm{d}x \qquad \text{(for } n \neq 1\text{)}$$

$$\int \sec^n x \, \mathrm{d}x = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} \int \sec^{n-2} x \, \mathrm{d}x^{[1]}$$

$$\int \frac{dx}{\sec x + 1} = x - \tan\frac{x}{2} + C$$

$$\int \frac{dx}{\sec x - 1} = -x - \cot\frac{x}{2} + C$$

Integrands involving only cosecant

$$\int \csc ax \, dx = \frac{1}{a}\ln|\csc ax - \cot ax| + C$$

$$\int \csc^2 x \, dx = -\cot x + C$$

$$\int \csc^n ax \, dx = -\frac{\csc^{n-1} ax \cos ax}{a(n-1)} + \frac{n-2}{n-1}\int \csc^{n-2} ax \, dx \qquad (\text{for } n \neq 1)$$

$$\int \frac{dx}{\csc x + 1} = x - \frac{2\sin\frac{x}{2}}{\cos\frac{x}{2} + \sin\frac{x}{2}} + C$$

$$\int \frac{dx}{\csc x - 1} = \frac{2\sin\frac{x}{2}}{\cos\frac{x}{2} - \sin\frac{x}{2}} - x + C$$

Integrands involving only cotangent

$$\int \cot ax \, dx = \frac{1}{a}\ln|\sin ax| + C$$

$$\int \cot^n ax \, dx = -\frac{1}{a(n-1)}\cot^{n-1} ax - \int \cot^{n-2} ax \, dx \qquad (\text{for } n \neq 1)$$

$$\int \frac{dx}{1 + \cot ax} = \int \frac{\tan ax \, dx}{\tan ax + 1}$$

$$\int \frac{dx}{1 - \cot ax} = \int \frac{\tan ax \, dx}{\tan ax - 1}$$

Integrands involving both sine and cosine

$$\int \frac{dx}{\cos ax \pm \sin ax} = \frac{1}{a\sqrt{2}}\ln\left|\tan\left(\frac{ax}{2} \pm \frac{\pi}{8}\right)\right| + C$$

$$\int \frac{dx}{(\cos ax \pm \sin ax)^2} = \frac{1}{2a}\tan\left(ax \mp \frac{\pi}{4}\right) + C$$

$$\int \frac{dx}{(\cos x + \sin x)^n} = \frac{1}{n-1}\left(\frac{\sin x - \cos x}{(\cos x + \sin x)^{n-1}} - 2(n-2)\int \frac{dx}{(\cos x + \sin x)^{n-2}}\right)$$

$$\int \frac{\cos ax \, dx}{\cos ax + \sin ax} = \frac{x}{2} + \frac{1}{2a}\ln|\sin ax + \cos ax| + C$$

$$\int \frac{\cos ax \, dx}{\cos ax - \sin ax} = \frac{x}{2} - \frac{1}{2a}\ln|\sin ax - \cos ax| + C$$

$$\int \frac{\sin ax \, dx}{\cos ax + \sin ax} = \frac{x}{2} - \frac{1}{2a}\ln|\sin ax + \cos ax| + C$$

$$\int \frac{\sin ax \, dx}{\cos ax - \sin ax} = -\frac{x}{2} - \frac{1}{2a}\ln|\sin ax - \cos ax| + C$$

$$\int \frac{\cos ax \, dx}{\sin ax(1 + \cos ax)} = -\frac{1}{4a}\tan^2\frac{ax}{2} + \frac{1}{2a}\ln\left|\tan\frac{ax}{2}\right| + C$$

$$\int \frac{\cos ax \, dx}{\sin ax(1 - \cos ax)} = -\frac{1}{4a}\cot^2\frac{ax}{2} - \frac{1}{2a}\ln\left|\tan\frac{ax}{2}\right| + C$$

$$\int \frac{\sin ax \; dx}{\cos ax(1+\sin ax)} = \frac{1}{4a}\cot^2\left(\frac{ax}{2}+\frac{\pi}{4}\right) + \frac{1}{2a}\ln\left|\tan\left(\frac{ax}{2}+\frac{\pi}{4}\right)\right| + C$$

$$\int \frac{\sin ax \; dx}{\cos ax(1-\sin ax)} = \frac{1}{4a}\tan^2\left(\frac{ax}{2}+\frac{\pi}{4}\right) - \frac{1}{2a}\ln\left|\tan\left(\frac{ax}{2}+\frac{\pi}{4}\right)\right| + C$$

$$\int \sin ax \cos ax \; dx = -\frac{1}{2a}\cos^2 ax + C$$

$$\int \sin a_1 x \cos a_2 x \; dx = -\frac{\cos((a_1-a_2)x)}{2(a_1-a_2)} - \frac{\cos((a_1+a_2)x)}{2(a_1+a_2)} + C \qquad (\text{for } |a_1| \neq |a_2|)$$

$$\int \sin^n ax \cos ax \; dx = \frac{1}{a(n+1)}\sin^{n+1} ax + C \qquad (\text{for } n \neq -1)$$

$$\int \sin ax \cos^n ax \; dx = -\frac{1}{a(n+1)}\cos^{n+1} ax + C \qquad (\text{for } n \neq -1)$$

$$\int \sin^n ax \cos^m ax \; dx = -\frac{\sin^{n-1} ax \cos^{m+1} ax}{a(n+m)} + \frac{n-1}{n+m}\int \sin^{n-2} ax \cos^m ax \; dx \qquad (\text{for } m, n > 0)$$

also: $$\int \sin^n ax \cos^m ax \; dx = \frac{\sin^{n+1} ax \cos^{m-1} ax}{a(n+m)} + \frac{m-1}{n+m}\int \sin^n ax \cos^{m-2} ax \; dx \qquad (\text{for } m, n > 0)$$

$$\int \frac{dx}{\sin ax \cos ax} = \frac{1}{a}\ln|\tan ax| + C$$

$$\int \frac{dx}{\sin ax \cos^n ax} = \frac{1}{a(n-1)\cos^{n-1} ax} + \int \frac{dx}{\sin ax \cos^{n-2} ax} \qquad (\text{for } n \neq 1)$$

$$\int \frac{dx}{\sin^n ax \cos ax} = -\frac{1}{a(n-1)\sin^{n-1} ax} + \int \frac{dx}{\sin^{n-2} ax \cos ax} \qquad (\text{for } n \neq 1)$$

$$\int \frac{\sin ax \; dx}{\cos^n ax} = \frac{1}{a(n-1)\cos^{n-1} ax} + C \qquad (\text{for } n \neq 1)$$

$$\int \frac{\sin^2 ax \; dx}{\cos ax} = -\frac{1}{a}\sin ax + \frac{1}{a}\ln\left|\tan\left(\frac{\pi}{4}+\frac{ax}{2}\right)\right| + C$$

$$\int \frac{\sin^2 ax \; dx}{\cos^n ax} = \frac{\sin ax}{a(n-1)\cos^{n-1} ax} - \frac{1}{n-1}\int \frac{dx}{\cos^{n-2} ax} \qquad (\text{for } n \neq 1)$$

$$\int \frac{\sin^n ax \; dx}{\cos ax} = -\frac{\sin^{n-1} ax}{a(n-1)} + \int \frac{\sin^{n-2} ax \; dx}{\cos ax} \qquad (\text{for } n \neq 1)$$

$$\int \frac{\sin^n ax \; dx}{\cos^m ax} = \frac{\sin^{n+1} ax}{a(m-1)\cos^{m-1} ax} - \frac{n-m+2}{m-1}\int \frac{\sin^n ax \; dx}{\cos^{m-2} ax} \qquad (\text{for } m \neq 1)$$

also: $$\int \frac{\sin^n ax \; dx}{\cos^m ax} = -\frac{\sin^{n-1} ax}{a(n-m)\cos^{m-1} ax} + \frac{n-1}{n-m}\int \frac{\sin^{n-2} ax \; dx}{\cos^m ax} \qquad (\text{for } m \neq n)$$

also: $$\int \frac{\sin^n ax \; dx}{\cos^m ax} = \frac{\sin^{n-1} ax}{a(m-1)\cos^{m-1} ax} - \frac{n-1}{m-1}\int \frac{\sin^{n-2} ax \; dx}{\cos^{m-2} ax} \qquad (\text{for } m \neq 1)$$

$$\int \frac{\cos ax \; dx}{\sin^n ax} = -\frac{1}{a(n-1)\sin^{n-1} ax} + C \qquad (\text{for } n \neq 1)$$

$$\int \frac{\cos^2 ax \; dx}{\sin ax} = \frac{1}{a}\left(\cos ax + \ln\left|\tan\frac{ax}{2}\right|\right) + C$$

$$\int \frac{\cos^2 ax \; dx}{\sin^n ax} = -\frac{1}{n-1}\left(\frac{\cos ax}{a\sin^{n-1} ax} + \int \frac{dx}{\sin^{n-2} ax}\right) \qquad (\text{for } n \neq 1)$$

$$\int \frac{\cos^n ax \; dx}{\sin^m ax} = -\frac{\cos^{n+1} ax}{a(m-1)\sin^{m-1} ax} - \frac{n-m-2}{m-1}\int \frac{\cos^n ax \; dx}{\sin^{m-2} ax} \qquad (\text{for } m \neq 1)$$

also: $$\int \frac{\cos^n ax \; dx}{\sin^m ax} = \frac{\cos^{n-1} ax}{a(n-m)\sin^{m-1} ax} + \frac{n-1}{n-m}\int \frac{\cos^{n-2} ax \; dx}{\sin^m ax} \qquad (\text{for } m \neq n)$$

also: $\displaystyle\int \frac{\cos^n ax \, \mathrm{d}x}{\sin^m ax} = -\frac{\cos^{n-1} ax}{a(m-1)\sin^{m-1} ax} - \frac{n-1}{m-1}\int \frac{\cos^{n-2} ax \, \mathrm{d}x}{\sin^{m-2} ax}$ (for $m \neq 1$)

Integrands involving both sine and tangent

$$\int \sin ax \tan ax \, \mathrm{d}x = \frac{1}{a}(\ln|\sec ax + \tan ax| - \sin ax) + C$$

$$\int \frac{\tan^n ax \, \mathrm{d}x}{\sin^2 ax} = \frac{1}{a(n-1)}\tan^{n-1}(ax) + C \qquad \text{(for } n \neq 1)$$

Integrands involving both cosine and tangent

$$\int \frac{\tan^n ax \, \mathrm{d}x}{\cos^2 ax} = \frac{1}{a(n+1)}\tan^{n+1} ax + C \qquad \text{(for } n \neq -1)$$

Integrals containing both sine and cotangent

$$\int \frac{\cot^n ax \, \mathrm{d}x}{\sin^2 ax} = -\frac{1}{a(n+1)}\cot^{n+1} ax + C \qquad \text{(for } n \neq -1)$$

Integrands involving both cosine and cotangent

$$\int \frac{\cot^n ax \, \mathrm{d}x}{\cos^2 ax} = \frac{1}{a(1-n)}\tan^{1-n} ax + C \qquad \text{(for } n \neq 1)$$

Integrals with symmetric limits

$$\int_{-c}^{c} \sin x \, \mathrm{d}x = 0$$
$$\int_{-c}^{c} \cos x \, \mathrm{d}x = 2\int_{0}^{c} \cos x \, \mathrm{d}x = 2\int_{-c}^{0} \cos x \, \mathrm{d}x = 2\sin c$$
$$\int_{-c}^{c} \tan x \, \mathrm{d}x = 0$$
$$\int_{-\frac{a}{2}}^{\frac{a}{2}} x^2 \cos^2 \frac{n\pi x}{a} \, \mathrm{d}x = \frac{a^3(n^2\pi^2 - 6)}{24n^2\pi^2} \qquad \text{(for } n = 1,3,5...)$$

References

1. ^ Stewart, James. Calculus: Early Transcendentals, 6th Edition. Thomson: 2008

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