Real Analysis MAA 6616
Lecture 13
The Lebesgue Integral of a Bounded Function Over a Set of Finite Measure
Lebesgue Integrable Functions

Let $\phi = \sum_{j=1}^{N} a_j \chi_{E_j}$ be any measurable simple function on a measurable set $E$ (so $\phi$ could take positive and negative values). Let $\phi^+ = \max(\phi, 0)$ and $\phi^- = \max(-\phi, 0)$. So that both $\phi^+$ and $\phi^-$ are nonnegative simple functions on $E$ and $\phi = \phi^+ - \phi^-$. Define the Lebesgue integral of $\phi$ as

$$\int_E \phi \, dx = \int_E \phi^+ \, dx - \int_E \phi^- \, dx = \sum_{j=1}^{N} a_j \, m(E_j).$$

Let $E$ be a measurable set with finite measure and let $f : E \rightarrow \mathbb{R}$ be a bounded function. Define the lower and upper Lebesgue integrals of $f$ over $E$ as:

$$\underline{\int}_E f(x) \, dx = \sup \left\{ \int_E \phi(x) \, dx : \phi \text{ simple function and } \phi \leq f \right\}$$

$$\overline{\int}_E f(x) \, dx = \inf \left\{ \int_B \psi(x) \, dx : \psi \text{ simple function and } \psi \geq f \right\}$$

Note that since $f$ is bounded, then whenever the simple functions $\phi$ and $\psi$ satisfy $\phi \leq f \leq \psi$ we have $\int_E \phi \, dx \leq \int_E \psi \, dx$ and it follows that $\underline{\int}_E f \, dx \leq \int_E f \, dx$.

A bounded function $f$ on a measurable set $E$ with $m(E) < \infty$ is said to be Lebesgue integrable if $\underline{\int}_E f \, dx = \overline{\int}_E f \, dx$. The common value is the Lebesgue integral of $f$ on $E$ and is denoted $\int_E f \, dx$.
The following theorem follows directly from the definitions of the Riemann and Lebesgue integrals.

**Theorem (1)**

Let $E \subset \mathbb{R}^q$ be a measurable set with finite measure and $f : E \rightarrow \mathbb{R}$ be a bounded function. If $f$ is Riemann integrable over $E$, then it is Lebesgue integrable over $E$.

**Remark (1)**

There exist Lebesgue integrable functions that are not Riemann integrable. For example, the Dirichlet function on $[0, 1]$ given by $f(x) = 1$ if $x$ is rational and $f(x) = 0$ if $x$ is irrational is not Riemann integrable (Lecture 12). However, since $f = \chi_E$ where $E = \mathbb{Q} \cap [0, 1]$ is measurable, we have $\int_{[0, 1]} f \, dx = m(E) = 0$.

**Theorem (2)**

Let $E \subset \mathbb{R}^q$ be a measurable set with finite measure and $f : E \rightarrow \mathbb{R}$ be a bounded and measurable function. Then $f$ is Lebesgue integrable.

**Proof.**

Let $\epsilon > 0$. It follows from the Simple Approximation Lemma (Lecture 10) that there exist simple functions $\phi_\epsilon$ and $\psi_\epsilon$ on $E$ such that $\phi_\epsilon \leq f \leq \psi_\epsilon$ and $\psi_\epsilon - \phi_\epsilon \leq \frac{\epsilon}{m(E)}$. Therefore

$$0 \leq \int_E f \, dx - \int_E f \, dx \leq \int_E \psi_\epsilon dx - \int_E \phi_\epsilon dx = \int_E (\psi_\epsilon - \phi_\epsilon) dx \leq \epsilon.$$

Since $\epsilon > 0$ is arbitrary we have $0 = \int_E f \, dx - \int_E f \, dx$ and $f$ is Lebesgue integrable.
Theorem (3)

Let \( f, g : E \rightarrow \mathbb{R} \) be bounded and measurable on \( E \) with \( m(E) < \infty \). Then

1. (Linearity) \( \int_E (af + bg) \, dx = a \int_E f \, dx + b \int_E g \, dx \) for any \( a, b \in \mathbb{R} \).

2. (Monotonicity) If \( f \leq g \) a.e. on \( E \), then \( \int_E f \, dx \leq \int_E g \, dx \).

Proof.

1. The function \( af + bg \) is measurable and bounded and so integrable. Observe that if \( S \subset \mathbb{R} \) is bounded and \( a > 0 \), then \( \sup(aS) = a \sup(S) \) and if \( a < 0 \), \( \sup(aS) = a \inf(S) \) where \( aS = \{x = as \in \mathbb{R} : s \in S\} \). We first show that \( \int_E af \, dx = a \int_E f \, dx \). Suppose \( a > 0 \). If \( \phi \) and \( \psi \) are simple functions such that \( \phi \leq af \leq \psi \), then \( \frac{\phi}{a} \) and \( \frac{\psi}{a} \) are simple functions and \( \frac{\phi}{a} \leq f \leq \frac{\psi}{a} \).

\[
\int_E f \, dx = a \inf \left\{ \int_E \psi \, dx : \psi \geq f \right\} = \inf \left\{ \int_E \tilde{\psi} \, dx : \tilde{\psi} \geq af \right\} = \int_E af \, dx
\]

For \( a < 0 \), we have \( \int_E f \, dx = a \inf \left\{ \int_E \psi \, dx : \psi \geq f \right\} = \sup \left\{ \int_E \tilde{\phi} \, dx : \tilde{\phi} \leq af \right\} = \int_E af \, dx \)

We are left to prove \( \int_E (f + g) \, dx = \int_E f dx + \int_E g dx \). Let \( \epsilon > 0 \). There are simple functions \( \phi_1 \) and \( \phi_2 \) such that \( \phi_1 \leq f, \phi_2 \leq g, \int_E \phi_1 \geq \int_E f dx + \frac{\epsilon}{2} \) and \( \int_E \phi_2 \geq \int_E 2dx + \frac{\epsilon}{2} \). Therefore

\[
\int_E (f + g) \, dx \geq \int_E (\phi_1 + \phi_2) \, dx = \int_E \phi_1 dx + \int_E \phi_1 dx = \int_E fdx + \int_E gdx + \epsilon.
\]

Similarly we can show by using upper simple functions that \( \int_E (f + g) \, dx \leq \int_E f dx + \int_E g dx + \epsilon \). Since \( \epsilon \) is arbitrary we have \( \int_E (f + g) \, dx = \int_E f dx + \int_E g dx \).
2. The function \( h = g - f \) is measurable and \( h \geq 0 \) a.e. on \( E \). Hence for any simple function \( \psi \geq h \) we have \( \psi \geq 0 \) a.e. and so \( \int_E \psi \, dx \geq 0 \). It follows that \( \int_E h \, dx \geq 0 \). The linearity of the integral implies that \( \int_E g \, dx \geq \int_E f \, dx \).

**Theorem (4)**

*Let \( E \subset \mathbb{R}^q \) be a measurable set with finite measure and \( f : E \rightarrow \mathbb{R} \) be a bounded and measurable function. Let \( A \subset E \) and \( B \subset E \) be disjoint and measurable. Then*

\[
\int_{A \cup B} f \, dx = \int_{A} f \, dx + \int_{B} f \, dx.
\]

**Proof.**

First note that if \( F \subset E \) is measurable and \( \phi \) is a simple function on \( E \), then \( \phi \chi_F \) is a simple function on \( F \) (and on \( E \)) and \( \int_F \phi \, dx = \int_E \phi \chi_F \, dx \). It follows that from this observation and the definition of the Lebesgue integral that for any bounded measurable function \( f \) we have \( \int_F f \, dx = \int_E f \chi_F \, dx \).

Since \( A \) and \( B \) are disjoint then \( \chi_{A \cup B} = \chi_A + \chi_B \). We have then

\[
\int_{A \cup B} f \, dx = \int_E f \chi_{A \cup B} \, dx = \int_E f (\chi_A + \chi_B) \, dx = \int_E f \chi_A \, dx + \int_E f \chi_B \, dx = \int_A f \, dx + \int_B f \, dx.
\]

\( \square \)
**Theorem (5)**

Let $E \subset \mathbb{R}^q$ be a measurable set with finite measure and $f : E \rightarrow \mathbb{R}$ be a bounded and measurable function. Then

$$\left| \int_E f \, dx \right| \leq \int_E |f| \, dx.$$ 

**Proof.**

The function $|f|$ is bounded and measurable and $-|f| \leq f \leq |f|$. Therefore $-\int_E |f| \, dx \leq \int_E f \, dx \leq \int_E |f| \, dx$. 

The following example shows that if $\{f_n\}_n$ is a sequence of measurable functions with $f_n \rightarrow f$, then the sequence of integrals $\int_E f_n \, dx$ might not converge to $\int_E f \, dx$. Consider the sequence of functions $f_n(x)$ on $(0, 1)$ given by $f_n(x) = n$ for $0 < x < 1/n$ and $f_n(x) = 0$ for $1/n \leq x < 1$.

The sequence $f_n$ converges pointwise to $f = 0$ but $\int_0^1 f_n \, dx = 1$ does not converge to $\int_0^1 0 \, dx = 0$. 
Convergence Theorems

**Theorem (6)**

Let \( \{f_n\} \) be a sequence of bounded measurable functions on a set \( E \) with finite measure. Suppose that \( \{f_n\} \) converges uniformly on \( E \) to a function \( f \). Then

\[
\lim_{n \to \infty} \int_E f_n \, dx = \int_E f \, dx.
\]

**Proof.**

Since the convergence is uniform and each \( f_n \) is bounded, then the limit \( f \) is bounded. Also \( f \) is measurable (since a pointwise limit of measurable functions is measurable). Hence \( f \) is integrable. Let \( \epsilon > 0 \). It follows from the uniform convergence that there exists \( N > 0 \) such that for every \( n > N \) we have

\[
|f_n - f| \leq \frac{\epsilon}{m(E)}
\]

on \( E \). Therefore,

\[
\left| \int_E f_n \, dx - \int_E f \, dx \right| = \left| \int_E (f_n - f) \, dx \right| \leq \int_E |f_n - f| \, dx \leq \epsilon.
\]

\( \square \)

**Theorem (7)**

Let \( \{f_n\} \) be a sequence of measurable functions on a set \( E \) with finite measure. Suppose that \( \{f_n\} \) is uniformly bounded (i.e. there exists \( M > 0 \) such that \( |f_n| \leq M \) for all \( n \in \mathbb{N} \)) and suppose that the sequence converges pointwise to \( f \) on \( E \). Then

\[
\lim_{n \to \infty} \int_E f_n \, dx = \int_E f \, dx.
\]
Proof.
First note that since \( f_n \to f \) pointwise and \( |f_n| \leq M \) for all \( M \), then \( |f| \leq M \). Also since a pointwise limit of measurable functions is measurable, then \( f \) is measurable and so integrable.

Let \( \epsilon > 0 \). It follows from Egorov’s Theorem that there exists a measurable set \( F \subset E \) with \( m(E \setminus F) < \frac{\epsilon}{4M} \) such that \( f_n \to f \) uniformly on \( F \). Let \( N \in \mathbb{N} \) such that \( |f_n - f| < \frac{\epsilon}{2m(E)} \) for all \( n > N \). We have

\[
\left| \int_E f_n \, dx - \int_E f \, dx \right| = \left| \int_E (f_n - f) \, dx \right|
= \left| \int_F (f_n - f) \, dx + \int_{E \setminus F} f_n \, dx - \int_{E \setminus F} f \, dx \right|
\leq \int_F |f_n - f| \, dx + \int_{E \setminus F} |f_n| \, dx + \int_{E \setminus F} |f| \, dx
\leq \frac{\epsilon}{2m(E)} m(F) + M m(E \setminus F) + M m(E \setminus F) \leq \epsilon
\]