Real Analysis MAA 6616
Lecture 22
Absolutely Continuous Functions
A function \( f : [a, b] \rightarrow \mathbb{R} \) is absolutely continuous on \([a, b]\) if for any \( \epsilon > 0 \) there exists \( \delta > 0 \) such that for any countable (finite or infinite) collection of non-overlapping intervals \( \{I_j = [a_j, b_j]\}_j \) in \([a, b]\) we have
\[
\sum_j (b_j - a_j) < \delta \implies \sum_j |f(b_j) - f(a_j)| < \epsilon.
\]

Denote by \( AC[a, b] \) the space of absolutely continuous functions on \([a, b]\). Note that if \( f \in AC[a, b] \), then \( f \) is (uniformly) continuous on \([a, b]\). However the converse is not true.

**Lemma (1)**

The Cantor-Lebesgue function \( \phi : [0, 1] \rightarrow [0, 1] \) is continuous increasing but it is not absolutely continuous.

**Proof.**

Recall that the Cantor function \( \phi \) is constant on each (removed) middle third interval in the construction of the Cantor set and that \( \phi \) is increasing and \( \phi(0) = 0, \phi(1) = 1 \).

At the first step in the construction of the Cantor set. We have the two remaining intervals \([a_1, b_1] = [0, 1/3]\) and \([a_2, b_2] = [2/3, 1]\) so that
\[
\sum_{j=1}^{2} (b_j - a_j) = \frac{2}{3} \quad \text{and} \quad \sum_{j=1}^{2} \phi(b_j) - \phi(a_j) = 1
\]

At the second step \( C_2 \) is the union of the \( 2^2 \) intervals of length \( 3^{-2} \): \([a_1, b_1] = [0, 1/9], [a_2, b_2] = [2/9, 3/9], [a_3, b_3] = [6/9, 7/9], \) and \([a_4, b_4] = [8/9, 9/9] \). Hence
\[
\sum_{j=1}^{2^2} (b_j - a_j) = \left( \frac{2}{3} \right)^2 \quad \text{and} \quad \sum_{j=1}^{2^2} \phi(b_j) - \phi(a_j) = 1
\]

In general at the \( n \)-th step we get \( C_n \) as the disjoint union of \( 2^n \) intervals \([a_j, b_j]\) each with length \( 3^{-n} \) so that
\[
\sum_{j=1}^{2^n} (b_j - a_j) = \left( \frac{2}{3} \right)^n \quad \text{and} \quad \sum_{j=1}^{2^n} \phi(b_j) - \phi(a_j) = 1
\]

It follows that for \( \epsilon = 1/2 \), the condition for absolute continuity does not hold since we have a collection of finitely many intervals with total measure \((2/3)^n\) which can be made as small as we which while total variation of \( \phi \) is 1.
Denote by $\text{Lip}[a, b]$ the space of Lipschitz function on $[a, b]$. That is $f \in \text{Lip}[a, b]$ if and only if there exists $c > 0$ such that $|f(x) - f(y)| \leq c |x - y|$ for all $x, y \in [a, b]$.

**Theorem (1)**

*A Lipschitz function on $[a, b]$ is absolutely continuous on $[a, b]$: $\text{Lip}[a, b] \subset \text{AC}[a, b]$*

**Proof.**

If $[a_j, b_j] \subset [a, b]$, then $|f(b_j) - f(a_j)| \leq c |b_j - a_j|$. Hence

$$
\sum_j |f(b_j) - f(a_j)| \leq c \sum_j |b_j - a_j|
$$

Therefore for $\epsilon > 0$, we can take $\delta = \epsilon / c$ for $f$ to satisfy the definition of absolute continuity.

There exist absolutely continuous functions that are not Lipschitz continuous as illustrated below.

The Function $f(x) = \sqrt{x}$ is in $\text{AC}[0, 1]$ but not in $\text{Lip}[0, 1]$. First we verify $\sqrt{x} \notin \text{Lip}[0, 1]$. If it were Lipschitz, then there would be $c > 0$ such that for every $0 \leq x < y \leq 1$, we would have $\sqrt{y} - \sqrt{x} \leq c(y - x)$. In particular for $x = 0$ we would have $\sqrt{y} \leq cy$ for all $y \in (0, 1)$. This means $1 \leq c \sqrt{y}$ for all $y > 0$ which is absurd.

However $f \in \text{Lip}[\alpha, 1]$, with Lipschitz constant $c = \frac{1}{2\sqrt{\alpha}}$ if $\alpha > 0$. Indeed for $\alpha \leq x < y \leq 1$, we have

$$
\sqrt{y} - \sqrt{x} = \frac{y - x}{\sqrt{y} + \sqrt{x}} \leq \frac{1}{2\sqrt{\alpha}} (y - x).
$$

Now we prove that $\sqrt{x} \in \text{AC}[0, 1]$. Given $\epsilon > 0$, let $\delta = \frac{\epsilon^2}{2}$. Let $\{I_k = [u_k, v_k]\}_k$ be a countable collection of non-overlapping intervals in $[0, 1]$ such that $\sum_k \ell(I_k) < \delta$. Consider the point $x_0 = \epsilon^2 / 4$, there exists at most one interval $I_{k_0}$ that contains $x_0$ in its interior. In which we split $I_{k_0}$ into two intervals $[u_{k_0}, x_0]$ and $[x_0, v_{k_0}]$. Let $C_1$ be the collection of intervals $I_k$ contained in $[0, x_0]$ and $C_2$ be the collection of intervals $I_k$ contained in $[x_0, 1]$. Using the fact that $\sqrt{x}$ is an increasing function, we have

$$
\sum_{k, I_k \in C_1} (\sqrt{v_k} - \sqrt{u_k}) \leq \sqrt{x_0} = \frac{\epsilon}{2}.
$$

Using the fact that $\sqrt{x} \in \text{Lip}[x_0, 1]$ with Lipschitz constant $c = 1/2\sqrt{x_0} = 1/\epsilon$, we have

$$
\sum_{k, I_k \in C_2} (\sqrt{v_k} - \sqrt{u_k}) \leq \frac{1}{\epsilon} \sum_{k, I_k \in C_2} \ell(I_k) \leq \frac{\epsilon}{2}.
$$

This shows that $\sqrt{x} \in \text{AC}[0, 1]$. \qed
Theorem (2)

An absolutely continuous function on $[a, b]$ is of bounded variation on $[a, b]$: $AC[a, b] \subset BV[a, b]$. Moreover, an absolutely continuous function can be written as the difference of two increasing absolutely continuous functions.

Proof.

Let $f \in AC[a, b]$, we need to show $f \in BV[a, b]$. Let $\epsilon = 1$ and $\delta > 0$ be a corresponding positive number so that $f$ satisfies the absolute continuity property for the pair $(\epsilon, \delta)$. Let $N \in \mathbb{N}$ be such that $N \geq \frac{b-a}{\delta}$ and for $i = 0, \ldots, N$ let $x_i = a + i \frac{b-a}{N}$ so $P = \{x_i\}_{i=0}^{N}$ be a partition of $[a, b]$ by equally spaced points and $x_{i+1} - x_i \leq \delta$. Now let $Q$ be any partition of $[a, b]$ and for every $i = 1, \ldots, N$, let $Q_i = (Q \cap [x_i-1, x_i]) \cup \{x_i-1, x_i\}$ so that $Q_i$ is a partition of the interval $[x_{i-1}, x_i]$. Since $x_i - x_{i-1} \leq \delta$, then $V(f_{[x_{i-1}, x_i]}, Q_i) \leq 1$ (absolute continuity of $f$ restricted to $[x_{i-1}, x_i]$).

Hence,

$$V(f, Q) \leq \sum_{i=1}^{N} V(f_{[x_{i-1}, x_i]}, Q_i) \leq \sum_{i=1}^{N} 1 = N.$$ 

Therefore $V(f, [a, b]) \leq N$ and $f \in BV[a, b]$.

Now we prove that $f$ can be written as the difference of two absolutely continuous and increasing functions. As was done earlier (Lecture 21), since $f \in BV[a, b]$ we can write $f$ as the difference of two increasing functions. Namely, $f = g - h$, where $h(x) = V(f, [a, x])$ and $g(x) = f(x) + V(f, [a, x])$. To complete the proof, we need to verify that $h$ is absolutely continuous. Let $\epsilon > 0$ and let $\delta > 0$ so that $f$ satisfies the absolute continuity property for the pair $\epsilon/2, \delta$. Let $\{I_k = [u_k, v_k]\}_{k=1}^{n}$ be a collection of disjoint subintervals of $[a, b]$ such that $\sum_k \ell(I_k) < \delta$. For $k \in \{1, \ldots, n\}$, let $P_k$ be a partition of the interval $I_k$. Then $\sum_{k=1}^{n} V(f_{I_k}, P_k) < \frac{\epsilon}{2}$. By taking the supremum over each partition $P_k$ of $I_k$, we get

$$\sum_{k=1}^{n} V(f_{I_k}) \leq \epsilon/2.$$ 

Since $V(f_{I_k}) = h(v_k) - h(u_k)$, we have proved

$$\sum_{k=1}^{n} (v_k - u_k) < \delta \implies \sum_{k=1}^{n} (h(v_k) - h(u_k)) < \epsilon.$$ 

That is $h \in AC[a, b]$. 


Let $f \in \mathcal{L}(a, b)$. Extend $f$ to the interval $[a, b + 1]$ by defining it as $f(x) = f(b)$ for any $x \in (b, b + 1]$. For any $0 < h < 1$ define the divided difference function $D_h f$ and the average function $A_n f$ on $[a, b]$ by

$$D_h f(x) = \frac{f(x + h) - f(x)}{h} \quad \text{and} \quad A_n f(x) = \frac{1}{h} \int_x^{x+h} f(s)ds.$$  

Note that if $[c, d] \subset [a, b]$, then $\int_c^d D_h f(x)dx = A_n f(d) - A_n f(c)$.  

Recall that a collection $C$ of measurable functions on a set $E$ is said to be uniformly integrable over $E$ if for every $\epsilon > 0$, there exists $\delta > 0$ such that for every $f \in C$ we have $\int_A |f| \, dx < \epsilon$ for all $A \subset E$ with $m(A) < \delta$.

**Theorem (3)**

Let $f$ be a continuous function on a closed and bounded interval $[a, b]$. Then $f$ is absolutely continuous on $[a, b]$ if and only if the family of divided difference functions $\{D_h f\}_{0 < h \leq 1}$ is uniformly integrable over $[a, b]$.

**Proof.**

"$\Longleftarrow$" Suppose that $\{D_h f\}_{0 < h \leq 1}$ is uniformly integrable over $[a, b]$. Let $\epsilon > 0$ and a corresponding $\delta > 0$ such that for every $h \in (0, 1]$ we have $\int_A |D_h f| \, dx < \epsilon$ whenever $A \subset [a, b]$ has measure $m(A) < \delta$.

Now let $\{I_k = (u_k, v_k)\}_{k=1}^n$ be a collection of disjoint intervals in $[a, b]$ such that $\sum_k \ell(I_k) < \delta$. Let $E = \bigcup_k I_k$. We have $m(E) < \delta$. For any $h \in (0, 1]$ and any index $k = 1, \ldots, n$ we have $A_n f(v_k) - A_n f(u_k) = \int_{u_k}^{v_k} D_h f(x)dx$.

Therefore for every $h \in (0, 1]$ we

$$\sum_{k=1}^n |A_n f(v_k) - A_n f(u_k)| \leq \sum_{k=1}^n \int_{I_k} |D_h f(x)| \, dx = \int_E |D_h f(x)| \, dx < \epsilon.$$  

Since $f$ is continuous, then $A_n f(x) \to f(x)$ as $h \to 0$, then it follows from the passage to the limit in the above inequality that $\sum_{k=1}^n |f(v_k) - f(u_k)| \leq \epsilon$ and $f \in \text{AC}[a, b]$. 

Proof.

CONTINUED:

"\Longrightarrow" Suppose \( f \in \text{AC}[a, b] \). Then we can express \( f \) as \( f = g - h \) with \( g, h \in \text{AC}[a, b] \) and increasing. We can therefore assume that \( f \) is increasing. This implies that the divided difference functions \( D_h f \) are nonnegative. Observe that if \([\alpha, \beta] \subset [a, b + 1]\) then, using change of variables, we have

\[
\int_\alpha^\beta D_h f(x) \, dx = \frac{1}{h} \left[ \int_{\alpha + h}^{\beta + h} f(s) \, ds - \int_\alpha^{\beta} f(s) \, ds \right] = \frac{1}{h} \int_0^h m(t) \, dt
\]

where \( m(t) = f(\beta + t) - f(\alpha + t) \).

Now we prove that \( \{D_h f\}_{0 < h \leq 1} \) is uniformly integrable over \([a, b]\). Let \( \epsilon > 0 \) and let \( \delta > 0 \) such that \( f \) satisfies the absolute continuity property for the pair \( \epsilon', \delta \) with \( \epsilon' < \epsilon \). Thus if \( \{I_k = (u_k, v_k)\}_{k=1}^n \) is a collection of disjoint intervals in \([a, b]\) such that \( \sum_k \ell(I_k) < \delta \), then \( \sum_k (f(v_k) - f(u_k)) < \epsilon' \). Note that for any \( 0 \leq t \leq 1 \), we have \( \sum_k \ell(t + I_k) < \delta \), and \( \sum_k (f(v_k + t) - f(u_k + t)) < \epsilon' \). Let \( U = \bigcup_k I_k \). Then \( m(U) < \delta \) and it follows from the above observations that

\[
\int_U D_h f(x) \, dx = \frac{1}{h} \int_0^h \sum_{k=1}^n (f(v_k + t) - f(u_k + t)) \, dt < \epsilon'
\]

Let \( E \subset [a, b] \) be such that \( m(E) < \delta / 2 \). There exists a \( G_\delta \) set \( G \) such that \( E \subset G \) and \( m(G) = m(E) \). The set \( G \) can be written as \( G = \bigcap_n U_n \) where \( \{U_n\}_n \) is nested collection of open set. Then there exists \( p \in \mathbb{N} \) such that \( E \subset U_p \) and \( m(U_p) < 2\delta / 3 \). Now \( U_p = \bigcup_k V_{k,p} \) where \( V_{k,p} \) is a disjoint union of a finite collection of open intervals and \( V_{k,p} \subset V_{k+1,p} \). Since \( m(V_{k,p}) < m(U_p) < 2\delta / 3 \), then it follows from the above estimate that

\[
\int_{V_{k,p}} D_h f(x) \, dx < \epsilon'
\]

for all \( k \). Hence

\[
\int_{U_p} D_h f(x) \, dx = \lim_{k \to \infty} \int_{V_{k,p}} D_h f(x) \, dx \leq \epsilon'.
\]

Finally

\[
\int_E D_h f(x) \, dx \leq \int_{U_p} D_h f(x) \, dx = \lim_{k \to \infty} \int_{V_{k,p}} D_h f(x) \, dx \leq \epsilon' < \epsilon.
\]
**Theorem (4)**

Let \( f \in AC[a, b] \). Then \( f \) is differentiable a.e. on \([a, b]\), \( f' \in \mathcal{L}(a, b) \), and
\[
\int_a^b f'(x)dx = f(b) - f(a).
\]

**Proof.**

Since \( f \in AC[a, b] \), then it follows from
\[
\int_a^b D_hf(x)dx = Av_hf(b) - Av_hf(a) = \frac{1}{h} \int_b^{b+h} f(x)dx - \int_a^{a+h} f(x)dx,
\]
by letting \( h \to 0 \) that
\[
\lim_{h \to 0} \frac{1}{h} \left[ \int_a^b (f(x+h) - f(x))dx \right] = f(b) - f(a).
\]

The function \( f \) can be written as \( f = g - h \) with \( g, h \in AC[a, b] \) increasing. Then \( f \) is differentiable a.e. and \( f' \in \mathcal{L}(a, b) \) (Corollary 1 in Lecture 21). Therefore \( D_{1/n}f \to f' \) pointwise a.e. in \([a, b]\). We know from Theorem 3 that the collection \( \{D_{1/n}f\}_n \) is uniformly integrable. Consequently the Vitali Convergence Theorem (Lecture 16) implies
\[
f(b) - f(a) = \lim_{n \to \infty} \int_a^b D_{1/n}f(x)dx = \int_a^b \lim_{n \to \infty} D_{1/n}f(x)dx = \int_a^b f'(x)dx
\]

Let \( g \in \mathcal{L}(a, b) \), the function \( f : [a, b] \to \mathbb{R} \) defined by
\[
f(x) = \lambda + \int_a^x g(t)dt,
\]
\( \lambda \in \mathbb{R} \) constant, is called an **indefinite integral** of \( g \) over \([a, b]\).

**Theorem (5)**

We have the following: \( f \in AC[a, b] \) if and only if \( f \) is an indefinite integral (of \( f' \)).
Proof.
"⇒" Let \( f \in \text{AC}[a, b] \), then \( f \) differentiable a.e. and \( f' \in \mathcal{L}(a, b) \). For any \( x \in [a, b] \) we have \( f \in \text{AC}[a, x] \) and Theorem 4 gives \( f(x) = f(a) + \int_a^x f'(t) \, dt \). Therefore \( f \) is an indefinite integral of \( f' \).

"⇐" Suppose that \( f \) is an indefinite integral of \( g \in \mathcal{L}(a, b) \): \( f(x) = f(a) + \int_a^x g(t) \, dt \).

Let \( \varepsilon > 0 \). It follows from \( |g| \in \mathcal{L}(a, b) \) that there exists \( \delta > 0 \) such that \( \int_E |g| \, dx < \varepsilon \) whenever \( E \subset [a, b] \) satisfies \( m(E) < \delta \) (Proposition 1, Lecture 17). Let \( \{I_k = (u_k, v_k)\}_{k=1}^n \) be a collection of disjoint open intervals in \([a, b] \) such that \( \sum_k \ell(I_k) < \delta \). Let \( E = \bigcup_k I_k \). Then

\[
\sum_{k=1}^n |f(v_k) - f(u_k)| = \sum_{k=1}^n \left| \int_{u_k}^{v_k} g(t) \, dt \right| \leq \sum_{k=1}^n \int_{u_k}^{v_k} |g(t)| \, dt = \int_E |g(t)| \, dt < \varepsilon
\]

Therefore \( f \in \text{AC}[a, b] \).

Corollary (1)

Let \( f : [a, b] \to \mathbb{R} \) be a monotone function. Then \( f \in \text{AC}[a, b] \) if and only if

\[
\int_a^b f'(x) \, dx = f(b) - f(a).
\]

Proof.

"⇒" This is a consequence of Theorem 5.

"⇐" Suppose that \( f \) is monotone (increasing). We know from Lebesgue’s Theorem that \( f \) is differentiable a.e. moreover

\[
\int_c^d f'(x) \, dx \leq f(d) - f(c) \quad \text{for all } a \leq c \leq d \leq b.
\]

Suppose further that \( \int_a^b f'(x) \, dx = f(b) - f(a) \). Let \( x \in [a, b] \). Then

\[
0 = \int_a^b f'(t) \, dt - (f(b) - f(a)) = \left[ \int_a^x f'(t) \, dt - (f(x) - f(a)) \right] + \left[ \int_x^b f'(t) \, dt - (f(b) - f(x)) \right] \leq 0
\]

This means \( f(x) = f(a) + \int_a^x f'(t) \, dt \) (\( f \) an indefinite integral) and consequently \( f \in \text{AC}[a, b] \) by Theorem 5.
Lemma (2)

Let \( f \in \mathcal{L}(a, b) \). Then \( f = 0 \) a.e. on \([a, b]\) if and only if \( \int_x^y f(t)dt = 0 \) for all \( a \leq x \leq y \leq b \).

Proof.

"\( \Leftarrow \)" Suppose that \( f \in \mathcal{L}(a, b) \) and \( \int_x^y f(t)dt = 0 \) for all \( a \leq x \leq y \leq b \). If \( U \subset [a, b] \) is open, then \( U = \bigcup_k I_k \), where \( \{I_k = (u_k, v_k)\}_k \) is a countable collection of disjoint intervals in \([a, b]\), then it follows from the additive property of the integral that \( \int_U f(x)dx = 0 \). Now if \( G \subset [a, b] \) is a \( G_\delta \) set, then we can write \( G = \bigcap_n U_n \) for some collection \( \{U_n\}_n \) of nested open sets in \([a, b]\). Then \( \int_G f(x)dx = \lim_{n \to \infty} \int_{U_n} f(x)dx = 0 \). Next, if \( E \subset [a, b] \) is an arbitrary measurable set, then there exists a \( G_\delta \) set \( G \) such that \( E \subset G \) and \( m(G \setminus E) = 0 \). Hence

\[
\int_E f(x)dx = \int_G f(x)dx - \int_{G \setminus E} f(x)dx = 0.
\]

Let \( f^+ = \max(f, 0) \) and \( f^- = \max(-f, 0) \) be the positive and negative parts of \( f \). Both are nonnegative integrable functions and \( f = f^+ - f^- \). Let

\[
E^+ = (f^+)^{-1}(\mathbb{R}) = \{x \in [a, b] : f(x) \geq 0\} \quad \text{and} \quad E^- = (f^-)^{-1}(\mathbb{R}) = \{x \in [a, b] : f(x) \leq 0\}.
\]

Then \( \int_a^b f^\pm(x)dx = \pm \int_{E^\pm} f(x)dx = 0 \). We know that if a nonnegative function has a vanishing integral, then the function is 0 a.e. Thus \( f^\pm = 0 \) a.e. and consequently \( f = f^+ - f^- = 0 \) a.e. on \([a, b] \). \( \square \)

Theorem (6)

Let \( f \in \mathcal{L}(a, b) \). Then \( \frac{d}{dx} \left[ \int_a^x f(t)dt \right] = f(x) \) for almost all \( x \in [a, b] \).
Proof.
Define the function $F$ on $[a, b]$ by $F(x) = \int_a^x f dt$. Then as an indefinite integral $F \in AC[a, b]$, $F$ is differentiable and $F' \in L([a, b])$ (Theorem 5). Now we need to verify that $F' - f = 0$ a.e. on $[a, b]$. For this it is enough to verify that if $a \leq x_1 < x_2 \leq b$, then $\int_{x_1}^{x_2} (F' - f) dt = 0$ (Lemma 2):

$$\int_{x_1}^{x_2} (F' - f) dt = \int_{x_1}^{x_2} F' dt - \int_{x_1}^{x_2} f dt = F(x_2) - F(x_1) - \int_{x_1}^{x_2} f dt = \int_a^{x_2} f dt - \int_a^{x_1} f dt - \int_{x_1}^{x_2} f dt = 0.$$

A function $s \in BV[a, b]$ is said to be singular if $s' = 0$ a.e. on $[a, b]$. The Cantor-Lebesgue function $\phi$ is an example of a nonconstant singular function. It follows from Theorem 4 that if an absolutely continuous is singular, then it is constant. The following theorem (Lebesgue) gives a decomposition of a function with bounded variation as the sum of an absolutely continuous function and a singular function.

**Theorem (7)**

Let $f \in BV[a, b]$. Then $f$ can be written as $f = g + h$, with $g \in AC[a, b]$ and $h$ a singular function.

**Proof.**

Since $f \in BV[a, b]$, then $f$ is differentiable a.e. and $f' \in L(a, b)$. Let $g = \int_a^x f' dt$ and $h = g - f$. Then $g \in AC[a, b]$, $h \in BV[a, b]$ and $g' = f'$ and $h' = 0$. 

\[\square\]