Real Analysis MAA 6616
Lecture 30
Hahn and Jordan Decompositions
Radon-Nikodym Theorem
Hahn Decomposition

**Theorem (1)**

Let \((X, \mathcal{A}, \mu)\) be a signed measure space. Then there exists positive and negative sets \(P, N \in \mathcal{A}\) such that \(P \cap N = \emptyset\) and \(P \cup N = X\). If \(\mu\) takes negative values, then \(\mu(N) < 0\) and if \(-\mu\) takes negative values \(\mu(P) > 0\). Moreover we have uniqueness of the such a pair in the sense that if \(P', N'\) is any other such pair, then the symmetric difference sets \(P \triangle P'\) and \(N \triangle N'\) are null sets.

**Proof.**

If there are no \(\mu\)-negative sets, then there is nothing to prove. Suppose then that there are negative sets. Let
\[
 r = \inf\{\mu(A) : A \text{ a negative set}\} \quad \text{and} \quad \{A_j\}_{j=1}^\infty \quad \text{be a sequence of negative sets such that} \quad \mu(A_j) \to r. 
\]
Define \(N = \bigcup_n A_n\) and \(P = X \setminus N\). Then \(N\) and \(P\) are disjoint, \(X = P \cup N\). We need to verify that \(N\) is negative and \(P\) positive.

Let \(B_1 = A_1, B_2 = A_2 \setminus B_1\), in general let \(B_k = A_k \setminus (B_1 \cup \cdots \cup B_{k-1})\). Note that since \(A_n\) is a negative set and \(B_n \subset A_n\), then \(\mu(B_n) \leq 0\); the \(B_n\)'s are disjoint; and \(\bigcup_n B_n = \bigcup_n A_n = N\). Let \(E \subset N\), then \(E \cap B_n \subset A_n\) and \(\mu(E \cap B_n) \leq 0\). We have \(E = \bigcup_n (E \cap B_n)\) and \(\mu(E) = \sum_n \mu(E \cap B_n) \leq 0\). This means that \(N\) is a negative set.

Moreover, we can write \(N = A_n \cup (N \setminus A_n)\) to get \(\mu(N) \leq \mu(A_n) + \mu(N \setminus A_n) \leq \mu(A_n)\). By letting \(n \to \infty\) we get \(\mu(N) \leq r\). Therefore \(\mu(N) = r\) and \(r > -\infty\).

Now we need to verify that \(P = X \setminus N\) is a positive set. By contradiction, if it were not, then there would exist \(C \subset P\) such that \(\mu(C) < 0\). By Proposition 2 (Lecture 29) there exists a negative set \(N_1 \subset N\). Let \(\tilde{N} = N \cup N_1\). Then \(\tilde{N}\) is a negative set and \(\mu(\tilde{N}) = \mu(N) + \mu(N_1) < r\) which is impossible. Therefore \(P\) is a positive set.

Now suppose that \(\mu\) is not a positive measure, we need to verify that \(\mu(N) < 0\). If \(\mu(N) = 0\). Let \(E \in \mathcal{A}\), then \(E = (E \cap N) \cup (E \cap P)\) and we have
\[
 \mu(E) = \mu(E \cap N) + \mu(E \cap P) \geq 0 + \mu(E \cap P) \geq 0.
\]
This means \(\mu\) is a positive measure (a contradiction). Similar argument can be used to verify that if \(-\mu\) is not a positive measure, then \(\mu(P) > 0\).

To verify uniqueness of the pair \(N, P\). Suppose \(N', P'\) is another such pair. Note that \(N \setminus N' = P' \setminus P\) so that if \(S \subset N \setminus N' = P' \setminus P\) then \(\mu(S) \leq 0\) as a subset of a negative set and \(\mu(S) \geq 0\) as a subset of a positive set. Therefore \(\mu(S) = 0\). Any set \(A \subset N \setminus N'\) can be written as \(A = S \cup T\) with \(S \subset N \setminus N'\) and \(T \subset N' \setminus N\). Therefore \(\mu(A) = \mu(S) + \mu(T) = 0\). □
Let \( f \in \mathcal{L}(\mathbb{R}^n, dm) \) \((m \text{ is the Lebesgue measure})\) and let \( \mathcal{M} \) be the \( \sigma \)-algebra of Lebesgue measurable sets in \( \mathbb{R}^n \). Defined a signed measure \( \mu \) on \( \mathcal{M} \) by \( \mu(E) = \int_E f dm \). Let \( P = \{ x \in \mathbb{R}^n : f(x) \geq 0 \} \) and \( N = \{ x \in \mathbb{R}^n : f(x) < 0 \} \). The pair \( P, N \) satisfies Proposition 1.

Two measures \( \mu \) and \( \nu \) defined on the same sigma algebra \( \mathcal{A} \) are said to be \textit{mutually singular} and denoted \( \mu \perp \nu \) if there exist disjoint sets \( A, B \in \mathcal{A} \) such that \( X = A \cup B, \mu(A) = 0 \) and \( \nu(B) = 0 \).

Example

This example will establish that the Lebesgue measure and the Lebesgue-Stieltjes measure generated by the Cantor-Lebesgue function are mutually singular.

First define the Lebesgue-Stieltjes measure. Let \( \alpha : \mathbb{R} \longrightarrow \mathbb{R} \) be an increasing function that is continuous from the right (i.e. \( \lim_{x \to c^+} \alpha(x) = \alpha(c) \) for all \( c \in \mathbb{R} \)). For \( a < b \) define \( \nu((a, b]) = \alpha(b) - \alpha(a) \). The \( \nu \) extends as a measure \( \nu : \mathcal{M} \longrightarrow [0, \infty] \), where \( \mathcal{M} \) is the \( \sigma \)-algebra of Lebesgue measurable sets in \( \mathbb{R} \). The measure \( \nu \) is called the \textit{Lebesgue-Stieltjes} measure generated by the function \( \alpha \).

Now consider the function \( \alpha \) given by \( \alpha(x) = 0 \) if \( x \leq 0 \), \( \alpha(x) = 1 \) if \( x \geq 1 \), and \( \alpha(x) = \phi(x) \), if \( 0 \leq x \leq 1 \), where \( \phi : [0, 1] \longrightarrow [0, 1] \) is the Cantor-Lebesgue function. Let \( \nu \) be the Lebesgue-Stieltjes measure generated by this function.

Note that since \( \alpha \) is constant on \((-\infty, 0]\) and on \([1, \infty)\), then \( \nu(E) = 0 \) if \( E \) is contained in \( \mathbb{R} \setminus [0, 1] \).

If \( m \) is the Lebesgue measure on \( \mathbb{R} \), then \( m \perp \nu \). Indeed let \( A = C \), where \( C \subset [0, 1] \) is the Cantor set, and let \( B = \mathbb{R} \setminus C \).

We already noted that \( m(C) = 0 \), we are left to verify that \( \nu(\mathbb{R} \setminus C) = 0 \). The set \( \mathbb{R} \setminus C \) is a union of disjoint intervals:

\[
\mathbb{R} \setminus C = (-\infty, 0) \cup \bigcup_{j=1}^{\infty} I_j \cup (1, \infty)
\]

where \( I_j \) are the open middle third intervals removed from \([0, 1]\) in the construction of the Cantor set. Since \( \phi \) is constant on each interval \( I_j \), then \( \nu(I_j) = 0 \). We already noted that \( \nu(-\infty, 0) = 0 \) and \( \nu(1, \infty) = 0 \). Therefore \( \nu(\mathbb{R} \setminus C) = 0 \) and the two measures are mutually singular.
The Jordan Decomposition Theorem

**Theorem (2)**

Let \( \mu : \mathcal{A} \rightarrow (-\infty, \infty] \) be a signed measure on a space \( X \). Then there exist positive measures \( \mu^+ : \mathcal{A} \rightarrow [0, \infty] \) and \( \mu^- : \mathcal{A} \rightarrow [0, \infty] \) with \( \mu^+ \perp \mu^- \) and such that \( \mu = \mu^+ - \mu^- \). Furthermore, this decomposition is unique.

**Proof.**

Let \( P, N \in \mathcal{A} \) be such that \( P \) is a positive set, \( N \) is a negative set for \( \mu \), \( P \cap N = \emptyset \) and \( X = P \cup N \) (Hahn Decomposition Theorem). Define \( \mu^\pm \) as follows. For \( A \in \mathcal{A} \) set \( \mu^+(A) = \mu(A \cap P) \) and \( \mu^-(A) = -\mu(A \cap N) \). Then both \( \mu^+ \) and \( \mu^- \) are positive measures and \( \mu = \mu^+ - \mu^- \). Moreover, we have\( \mu^+(N) = \mu(P \cap N) = 0 \) and \( \mu^- (P) = \mu(P \cap N) = 0 \) so that \( \mu^+ \perp \mu^- \).

Now suppose that \( \mu = \nu^+ - \nu^- \) is another such decomposition with \( \nu^+ \perp \nu^- \). Let \( S^+, S^- \in \mathcal{A} \) be the associated pair in the Hahn decomposition: \( S^+ \cap S^- = \emptyset \), \( X = S^+ \cup S^- \), \( \nu^\pm (S^\mp) = 0 \). It follows from the uniqueness of the Hahn decomposition that \( P \triangle S^+ \) and \( N \triangle S^- \) are \( \mu \)-null sets.

Let \( A \in \mathcal{A} \). We have
\[
\nu^+(A) = \nu^+(A \cap S^+) = \nu^+(A \cap S^+) - \nu^-(A \cap S^+) = \mu(A \cap S^+) = \mu(A \cap P) = \mu^+(A).
\]

This shows that \( \mu^+ = \nu^+ \). A similar argument can be used to prove \( \mu^- = \nu^- \). \( \square \)

The measure \( |\mu| : \mathcal{A} \rightarrow [0, \infty] \) given by \( |\mu| = \mu^+ + \mu^- \) is called the variation measure of \( \mu \) and \( |\mu|(A) = \mu^+(A) + \mu^-(A) \) the total variation of \( A \).
Absolute continuous measures

Let $\mu$, $\nu$ be two measures defined in a $\sigma$-algebra over a set $X$. The measure $\nu$ is said to be absolutely continuous with respect to $\mu$, if every $\mu$-null set is also a $\nu$-null set. That is
\[
\forall A \in \mathcal{A} \quad \mu(A) = 0 \implies \nu(A) = 0.
\]
In this case we write $\nu \ll \mu$.

**Proposition (1)**

Let $\mu$ and $\nu$ be measures defined in $\mathcal{A}$ over $X$ and such that $\nu$ is finite ($\nu(X) < \infty$). Then $\nu \ll \mu$ if and only if for every $\epsilon > 0$ there exists $\delta > 0$ such that for every $A \in \mathcal{A}$ with $\mu(A) < \delta$, we have $\nu(A) < \epsilon$.

**Proof.**

"$\Leftarrow$" Let $A \in \mathcal{A}$ such that $\mu(A) = 0$, we need to show that $\nu(A) = 0$. Let $\epsilon > 0$ and $\delta > 0$ such that satisfies the condition of the proposition. Since $\mu(A) = 0 < \delta$, then $\nu(A) < \epsilon$. Since $\epsilon > 0$ is arbitrary, then $\nu(A) = 0$.

"$\Rightarrow$" Suppose that $\nu \ll \mu$. By contradiction, suppose that there exists $\epsilon_0 > 0$ such that for every $n \in \mathbb{N}$, there exists a set $A_n \in \mathcal{A}$ such that $\mu(A_n) < 2^{-n}$ and $\nu(A_n) > \epsilon_0$. Let $A = \limsup_n A_n = \bigcap_{n \geq 1} \bigcup_{k \geq n} A_k$. Then
\[
\mu(A) = \lim_{n \to \infty} \mu \left( \bigcup_{k \geq n} A_k \right) \leq \lim_{n \to \infty} \sum_{k=n}^{\infty} \mu(A_k) \leq \lim_{n \to \infty} \sum_{k=n}^{\infty} \frac{1}{2^k} = 0.
\]
This means $\mu(A) = 0$ and $\nu(A) \geq \epsilon_0 > 0$ which is a contradiction. \(\square\)

**Proposition (2)**

Let $\mu$ and $\nu$ be finite positive measures defined in $\mathcal{A}$ over $X$. Then either $\mu \perp \nu$ or else there exists $\epsilon > 0$ and a set $P \in \mathcal{A}$ such that $\mu(P) > 0$ and $P$ is a positive set for the measure $\nu - \epsilon \mu$. 
Proof.
Let $n \in \mathbb{N}$ and consider the signed measure $\nu - \frac{1}{n} \mu$. It follows from the Hahn decomposition that there exists a pair sets $P_n$ (positive) and $N_n$ (negative) such that $P_n \cap N_n = \emptyset$, $P_n \cup N_n = X$ for this measure.
Let $N = \bigcap_n N_n$ and $P = \bigcup_n P_n$. We have $X \setminus N = \bigcap_n (X \setminus N_n) = \bigcup_n P_n = P$. For each $n$ we have $N \subset N_n$ and
$$0 \leq \nu(N) \leq \nu(N_n) \leq \frac{1}{n} \mu(N_n) \leq \frac{1}{n} \mu(X).$$
Since $\mu(X) < \infty$, then $\nu(N) = 0$. Now we have two possibilities. Either $\mu(P) = 0$ and then $\nu \perp \mu$ or else $\mu(P) > 0$. In this case there exists $n_0$ such that $\mu(P_{n_0}) > 0$. Let $\epsilon = 1/n_0$. Then from the definition of $P_{n_0}$ we have
$$\nu(P_{n_0}) - \epsilon \mu(P_{n_0}) > 0.$$  

Remark (1)
We know from earlier examples that if $f$ is a nonnegative $\mu$-integrable function over $X$, then the set function $\nu : \mathcal{A} \longrightarrow \mathbb{R}^+$ given by $\nu(A) = \int_A f d\mu$ is a measure. The Radon-Nikodym gives a sufficient condition under which $\nu$ has the above form. The function $f$ is called the derivative (or density) of $\nu$ with respect to $\mu$ and is denoted $f = \frac{d\nu}{d\mu}$ or $d\nu = fd\mu$. 

\qed
Radon-Nikodym Theorem

**Theorem (3)**

Let $\mu : \mathcal{A} \rightarrow [0, \infty]$ be a $\sigma$-finite positive measure on $X$ and let $\nu : \mathcal{A} \rightarrow [0, \infty)$ be a finite measure on $X$ such that $\nu$ is absolutely continuous with respect to $\mu$ ($\nu \ll \mu$). Then there exists a $\mu$-integrable nonnegative function $f \in L(X, \mu)$ such that

$$\nu(A) = \int_A f \, d\mu \quad \text{for all } A \in \mathcal{A}.$$ 

Moreover if $g \in L(X, \mu)$ is any other such function, then $g = f$ a.e. in $X$.

**Proof.**

Uniqueness. If $f, g \in L(X, \mu)$ are two such functions, then $h = f - g \in L(X, \mu)$ and for every $A \in \mathcal{A}$ we have

$$\int_A hd\mu = \int_A (f - g) d\mu = \int_A fd\mu - \int_A gd\mu = \nu(A) - \nu(A) = 0$$

Hence $h = 0$ a.e. in $X$.

To prove the existence of the function $f$ we consider two cases.

- **Case 1:** $\mu$ is a finite measure ($\mu(X) < \infty$). Consider the family of functions

$$\mathcal{F} = \left\{ h \in L(X, \mu) : h \geq 0 \text{ and } \int_A hd\mu \leq \nu(A) \text{ for all } A \in \mathcal{A} \right\}$$

Note that $\mathcal{F} \neq \emptyset$ since $0 \in \mathcal{F}$ and if $\alpha, \beta \in \mathcal{F}$ then $\max(\alpha, \beta) \in \mathcal{F}$. Indeed, consider the sets $C = \{x : \alpha(x) \geq \beta(x)\}$ and $D = \{x : \alpha(x) < \beta(x)\} = X \setminus C$ so that $C \cap D = \emptyset$ and $C \cup D = X$. Let $A \in \mathcal{A}$. Then

$$\int_A \max(\alpha, \beta) d\mu = \int_{A \cap C} \alpha d\mu + \int_{A \cap D} \beta d\mu \leq \nu(A \cap C) + \nu(A \cap D) = \nu(A)$$

This implies $\max(\alpha, \beta) \in \mathcal{F}$. 

$\square$
Proof. CONTINUED

Let \( M = \sup \{ \int_X h \, d\mu : h \in F \} \). Note that since \( \int_X h \, d\mu \leq \nu(X) < \infty \), then \( M < \infty \). Let \( \{h_n\}_n \subset F \) such that \( \lim_{n \to \infty} \int_X h_n \, d\mu = M \). For each \( n \in \mathbb{N} \), let \( f_n = \max(h_1, \ldots, h_n) \). Then \( f_n \in F \) and \( \{f_n\}_n \) is an increasing sequence. Let \( f = \lim_n f_n \). It follows from the Monotone Convergence Theorem that for every \( A \in \mathcal{A} \) we have

\[
\int_A f \, d\mu = \lim_{n \to \infty} \int_A f_n \, d\mu \leq \nu(A).
\]

Hence \( f \in F \). Moreover

\[
\int_X f \, d\mu = \lim_{n \to \infty} \int_X f_n \, d\mu \geq \lim_{n \to \infty} \int_X h_n \, d\mu = M.
\]

Therefore \( \int_X f \, d\mu = M \).

We are left to verify that \( \int_A f \, d\mu = \nu(A) \). For this we define a positive measure \( \lambda : \mathcal{A} \to [0, \infty) \) by

\[
\lambda(A) = \nu(A) - \int_A f \, d\mu.
\]

We claim that \( \lambda \perp \mu \). By contradiction, if \( \lambda \) were not mutually singular to \( \mu \), then (Proposition 2) there exists \( \epsilon > 0 \) and \( P \in \mathcal{A} \) such that \( \mu(P) > 0 \) and \( P \) is a positive set for the measure \( \lambda - \epsilon \mu \). Let \( A \in \mathcal{A} \). We have

\[
\nu(A) - \int_A f \, d\mu = \lambda(A) \geq \lambda(A \cap P) \geq \epsilon \mu(A \cap P) = \int_A \epsilon \chi_P \, d\mu.
\]

Then \( \nu(A) \geq \int_A (f + \epsilon \chi_P) \, d\mu \). This means \( h = f + \epsilon \chi_P \in F \). But,

\[
\int_X h \, d\mu = \int_X f \, d\mu + \epsilon \int_X \chi_P \, d\mu = M + \epsilon \mu(P) > M.
\]

This is a contradiction and so \( \lambda \perp \mu \).

\[\square\]
Proof.

CONTINUED

Since \( \lambda \perp \mu \), there exists \( Z \in \mathcal{A} \) such that \( \mu(Z) = 0 \) and \( \lambda(X \setminus Z) = 0 \). It follows from \( \nu \ll \mu \) that \( \nu(Z) = 0 \). Therefore \( \lambda(Z) = \nu(Z) - \int_Z f d\mu = 0 \). We have then \( \lambda(X) = \lambda(Z) + \lambda(X \setminus Z) = 0 \). So that \( \lambda = 0 \). Therefore \( \nu(A) = \int_A f d\mu \) for \( A \in \mathcal{A} \). This completes the proof when \( \mu \) is finite.

Case 2: \( \mu \) is \( \sigma \)-finite. In this case there exists a sequence \( \{X_j\}_j \subset \mathcal{A} \) such that \( X_j \nearrow X \) and \( \mu(X_j) < \infty \) for all \( j \).

For \( j \in \mathbb{N} \), Let \( \mu_j \) and \( \nu_j \) be the restrictions of \( \mu \) and \( \nu \) to \( X_j \): \( \mu_j(A) = \mu(A \cap X_j) \) and \( \nu_j(A) = \nu(A \cap X_j) \) for \( A \in \mathcal{A} \). Then \( \mu_j \) is a finite measure and \( \nu_j \ll \mu_j \). Indeed, if \( \mu_j(E) = 0 \), then \( \mu(E \cap X_j) = 0 \) and so \( 0 = \nu(E \cap X_j) = \nu_j(E) \).

It follows from the first case that there exists \( f_j \in \mathcal{L}(X, \mu_j) \) such that \( d\mu_j = f_j d\mu_j \). It follows from the uniqueness of the density that \( f_i = f_j \) a.e. in \( X \) if \( i \leq j \). Define \( f \in \mathcal{L}(X, \mu) \) by \( f(x) = f_j(x) \) if \( x \in X_j \).

Let \( A \in \mathcal{A} \), we have

\[
\nu(A) = \lim_{j \to \infty} \nu(A \cap X_j) = \lim_{j \to \infty} \nu_j(A) = \lim_{j \to \infty} \int_A f_j d\mu_j = \lim_{j \to \infty} \int_{A \cap X_j} f d\mu = \int_A f d\mu.
\]

This completes the proof. \( \square \)

Theorem (4)

Let \( \mu \) and \( \nu \) be respectively \( \sigma \)-finite and finite positive measures defined over a \( \sigma \)-algebra \( \mathcal{A} \) of a space \( X \). Then there exist unique positive measures \( \lambda \) and \( \rho \) such that \( \nu = \lambda + \rho \), with \( \rho \ll \mu \) and \( \lambda \perp \mu \).

Remark (2)

This theorem is known as the Lebesgue Decomposition Theorem. Its proof is analogous to that of the Radon-Nikodym Theorem. What is missing here is \( \nu \ll \mu \). Once \( \mathcal{F}, M \), and \( f \) are defined exactly as in the proof of the Radon-Nikodym Theorem, we can define \( \rho \) by

\[
\rho(A) = \int_A f d\mu \text{ and } \lambda = \nu - \rho.
\]