MHF 3404 - HISTORY OF MATHEMATICS — FLORIDA INT'L UNIV.
TEST #2 - Spring '09 — TIME: 75 min.

Answer all 10 questions. Provide all reasoning and show all working. An unjustified answer will receive little or no credit. Begin each question on a separate page. Parts (a) and (b) should be done on the same page if possible. You may write on both sides of the paper.

(10) 1. If 2 ladies can sew 5 dresses in 3 hours, how long will it take 7 ladies to sew 9 dresses? (Explain how you got your answer.)

(10) 2. (a) Determine if 4839615 is divisible by 11 by using the Divisibility Test for 11.
(b) Determine if 5849613 is divisible by 7 by using the Divisibility Test for 7.

(10) 3. Find the solution to the difference equation $a_{n+2} - a_{n+1} - 6.a_n = 0$ with the initial conditions $a_0 = 5$ and $a_1 = 0$.

(10) 4. Express $\sin^2 x$ and $\cos^2 x$ in terms of $\cos(2x)$ by using the De Moivre-Euler formula and the formulas for sin(x) and cos(x) which involves complex exponentiation .

(12) 5. (a) Draw the part of the unit circle that is in the first quadrant of the coordinate system. Then draw the appropriate lines to show the values of the six trigonometric functions for a given angle A. (You must identify each of the six trigonometric function with the length of the corresponding line segment.)
(b) Starting with $\sin^2 x + \cos^2 x = 1$ and $\tan(x) = \sin(x)/\cos(x)$, find a formula expressing sin(x) in terms of tan(x) and constants only.

(12) 6. Let b and a be fixed positive real numbers; and x be any positive real number. By using the laws of indices and the fact that $\log_b(x)$ is the inverse of the function b^x, prove that
(a) $\log_b(1/x) = -\log_b(x)$ (b) $\log_a(x) = \log_b(x) / \log_b(a)$.

(12) 7. (a) Let $C(n,k) = (n!) / [(k!).(n-k)!]$. Prove that $C(n,k) = C(n-1,k-1) + C(n-1,k)$ combinatorially.
(b) Prove that the number of walks of length n+k from the point (0,0) to the point (n,k) on the coordinate grid is C(n+k, n).

(12) 8. (a) Explain why Giralamo Cardano was so anxious to extract from Niccolo Fontana (Tartaglia) the method of solving the cubic equation.
(b) Determine the nature of the solutions of the cubic equation $y^3 - 15y + 16 = 0$.

(12) 9. (a) Find the resolvant of the quartic equation $y^4 - (1/2).y^2 + 2.y + (1/16) = 0$.
(b) Using the resolvant, determine the nature of the solutions of this quartic equation.

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SOLUTIONS TO TEST # 2 — SPRING 2009

1. 2 ladies can sew 5 dresses in 3 hours.

$\therefore$ 1 lady will sew 5 dresses in 3×2 hours.

$\therefore$ 1 lady will sew 1 dress in $(3 \times 2)/5$ hours.

$\therefore$ 7 ladies will sew 1 dress in $(3 \times 2)/(5 \times 7)$ hours.

$\therefore$ 7 ladies will sew 9 dresses in $(3 \times 2 \times 9)/(5 \times 7)$ hours,

i.e., $54/35 = 1\frac{19}{35}$ hours.

2(a) $4 - 8 + 3 - 9 + 6 - 1 + 5 = (4+3+6+5) - (8+9+1) = 0.$

So 4839615 is divisible by 11

(b) 5849613 is divisible by 7

$\Leftrightarrow$ $584961 - 2(3) = 584955$ is divisible by 7

$\Leftrightarrow$ $58495 - 2(5) = 58485$ is divisible by 7

$\Leftrightarrow$ $5848 - 2(5) = 5838$ is divisible by 7

$\Leftrightarrow$ $583 - 2(8) = 567$ is divisible by 7

$\Leftrightarrow$ $56 - 2(7) = 42$ is divisible by 7.

Since 42 is divisible by 7, 5849613 is divisible by 7

3. $a_{n+2} - a_{n+1} - 6 \cdot a_n = 0$, $a_0 = 5$ and $a_1 = 0$

$\therefore (E^2 - E - 6)\, a_n = 0 \Rightarrow (E-3)(E+2) = 0$

$\therefore E = 3$ or -2. $\therefore a_n = A.(3)^n + B.(-2)^n$

$a_0 = 5 \Rightarrow 5 = A.(3)^0 + B.(-2)^0 \Rightarrow A + B = 5$

$a_1 = 0 \Rightarrow 0 = A.(3)^1 + B(-2) \quad 3A - 2B = 0$

$\therefore B = 3A/2$. $\therefore A + (3A)/2 = 5 \Rightarrow (5A)/2 = 5$

$\therefore A = 2$. Thus $B = 3A/2 = 3(2)/2 = 3$.

So $a_n = A.(3)^n + B.(-2)^n = 2.(3)^n + 3.(-2)^n$.

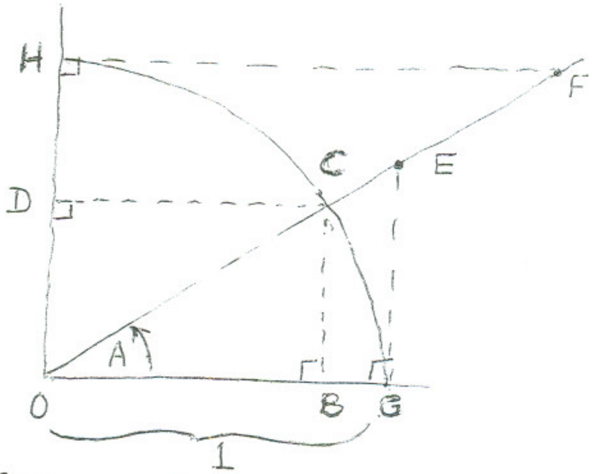
4 (a) $\cos^2 x = \left(\frac{e^{ix}+e^{-ix}}{2}\right)^2 = \frac{(e^{ix})^2 + 2e^{ix}.e^{-ix} + (e^{-ix})^2}{4}$

$= \frac{2}{4} + \frac{1}{2}\frac{e^{i2x}+e^{-i.2x}}{2} = \frac{1}{2} + \frac{1}{2}\cos(2x).$

(b) $\sin^2 x = \left(\frac{e^{ix}-e^{-ix}}{2i}\right)^2 = \frac{(e^{ix})^2 - 2e^{ix}.e^{-ix} + (e^{-ix})^2}{-4}$

$= \frac{-2}{-4} - \frac{1}{2}\cdot\frac{e^{i.2x}+e^{-i.2x}}{2} = \frac{1}{2} - \frac{1}{2}\cos(2x).$

5 (a)



$\sin(A) = \overline{BC}$

$\cos(A) = \overline{DC}$

$\tan(A) = \overline{GE}$

$\cot(A) = \overline{HF}$

$\sec(A) = \overline{OE}$

$\csc(A) = \overline{OF}$

(b) $\sin^2(x) + \cos^2(x) = 1$, so $\frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$

and hence $\tan^2(x) + 1 = 1/\cos^2(x)$. $\therefore \cos^2(x) = \frac{1}{1+\tan^2 x}$

Since $\tan x = \sin x/\cos x$, we get

$\sin x = \tan x . \cos x = \tan x . \sqrt{\frac{1}{1+\tan^2 x}} = \frac{\tan x}{\sqrt{1+\tan^2 x}}$

6 (a) Let $p = \log_b(x)$. Then $x = b^p$. So $1/x = 1/(b^p) = b^{-p}$

$\therefore \log_b(1/x) = \log_b(b^{-p}) = -p = -\log_b(x)$.

(b) Let $p = \log_b(x)$, $q = \log_b(a)$, and $s = \log_a(x)$. Then

$x = b^p$, $a = b^q$, and $x = a^s$

$\therefore x = a^s = (b^q)^s = b^{q.s}$. But $x = b^p$.

Hence $p = q.s$. So $s = p/q$.

$\therefore \log_a(x) = \log_b(x) / \log_b(a)$

7 (a) $C(n,k)$ = no. of subsets of $\{1,2,3,\dots,n\}$ with k elements

= no. of subsets of $\{1,2,3,\dots,n\}$ with k elements which include $\underline{1}$

+ no. of subsets of $\{1,2,3,\dots,n\}$ with k elements which exclude 1

7(a) $=$ no. of subsets of $\{2,3,\dots,n\}$ with $k-1$ elements
+ no. of subsets of $\{2,3,\dots,n\}$ with k elements
$= C(n-1,k-1) + C(n-1,k)$.

(b) Number of walks of length $n+k$ from $(0,0)$ to (n,k)
= no. of ways of arranging n R's and k U's in a row

$$= \frac{(n+k)!}{(n!)(k!)} = \frac{(n+k)!}{(n!)(n+k-k)!} = C(n+k,n).$$

8(a) Cardano was anxious to extract from Fontana, the method of solving the cubic for 3 reasons. First, he was naturally curious because he couldn't solve the cubic. Next, his student, Ferrari, reduce the solution of the quartic to the cubic. Finally, he thought it would make quite a "splash" if he were to publish it in a book, "The Great Art", that he was thinking of writing. [By the way Cardano gave credit to the others]

(b) $y^3 - 15y + 16 = 0$. $\therefore D = (P/3)^3 + (q/2)^2 = (-15/3)^3 + (16/2)^2$
$= -125 + 64 = -61 < 0$. So eq. has 3 real roots.

9(a) $y^4 - (1/2).y^2 + 2.y + (1/16) = 0$. So the resolvant is
$z^3 + 2p.z^2 + (p^2 - 4r).z - q^2 = 0$, i.e.
$z^3 - z^2 - 4 = 0$. ..... $(*)$

(b) Let $w = z - 1/3$. Then $z = w + 1/3$. So $(*)$ becomes
$(w + 1/3)^3 - (w + 1/3)^2 - 4 = 0$
$\therefore w^3 + 3.w^2\frac{1}{3} + 3.w.\left(\frac{1}{3}\right)^2 + \frac{1}{27} - \left[w^2 + 2.w.\frac{1}{3} + \left(\frac{1}{3}\right)^2\right] - 4 = 0$
$\therefore w^3 - \frac{1}{3}w - 4 - \frac{2}{27} = 0$. $\therefore w^3 - \frac{1}{3}w - \frac{110}{27} = 0$

Now
$$D = (P/3)^3 + (Q/2)^2 = (-1/9)^3 + \left(\frac{55}{27}\right)^2 > 0$$

So the resolvant $(*)$ has one real root & a pair of complex conjugate roots. $\therefore$ the quartic has 2 real roots and a pair of complex conjugate roots. [We can also solve $(*)$ diretly to get $(z-2)(z^2+z+2)=0$. $\therefore z = 2$ or $(-1 \pm i\sqrt{7})/2$.]