

Answer all 6 questions. No calculators, formula sheets or cell-phones are allowed. Provide all reasoning and show all working. An unjustified answer will receive little or no credit. Begin each of the 6 questions on 6 separate pages.

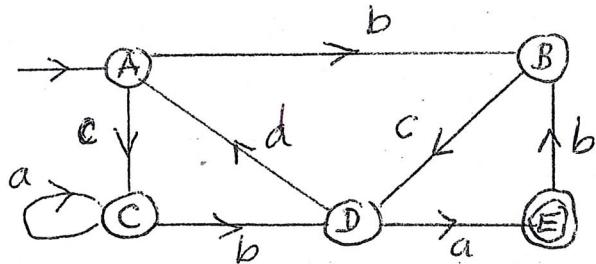
- (16) 1. (a) Find an NFA,  $M$ , which is equivalent to the RLG  $G$  given below.

$$G: \rightarrow A, \quad A \rightarrow 10A, \quad A \rightarrow 0B, \quad A \rightarrow 1C, \quad B \rightarrow 0C, \quad B \rightarrow 0D, \\ C \rightarrow 00, \quad C \rightarrow 10E, \quad D \rightarrow 1A, \quad D \rightarrow \lambda, \quad E \rightarrow B.$$

- (b) Find an RLG,  $G$ , which is equivalent to the NFA in Problem 2(a) below.

- (16) 2. (a) Find a regular expression for the language accepted by the NFA  $M$  shown on the right.

- (b) Define what is the busy-beaver function,  $\beta(n)$ .



- (16) 3. (a) Define the initial functions and the operation called primitive recursion.

- (b) Show that  $f(x,y) = 2x+3y+1$  is a primitive recursive function by finding primitive recursive functions  $g$  and  $h$  such that  $f = \text{prec}(g,h)$ .

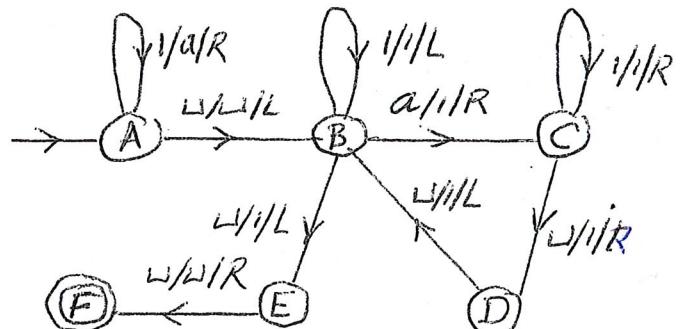
- (16) 4.(a) Define what is a  $\mu$ -recursive function.

- (b) Let  $f(x) = \text{Ceiling function of } [(2x+1)^{1/2}]$ . Show that  $f$  is a  $\mu$ -recursive function. [You may use the fact that PRED, MONUS, ADD, MULT & SIGN are primitive recursive if needed in #4, but you are not allowed to do so in Question #3.]

- (18) 5. (a) Define what it means for a relation  $R$  on  $\mathbb{N}^k$  to be Turing-decidable.

- (b) Show what happens at each step if (i) 1 and (ii)  $\lambda$  are the inputs for the TM,  $M$ , shown on the right.

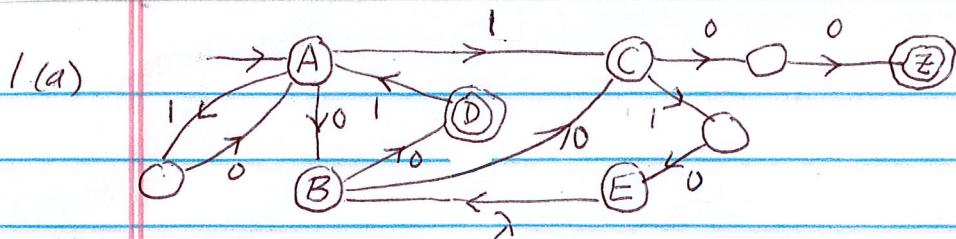
- (c) What is the function computed by  $M$  in monadic (base 1) notation?



- (18) 6. Determine which of the following languages are regular and which are not.

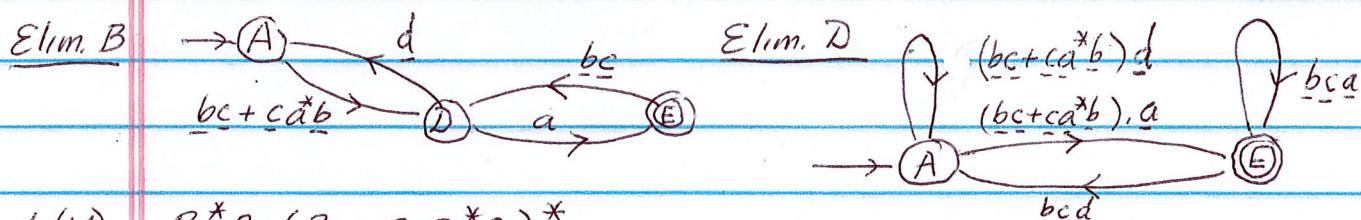
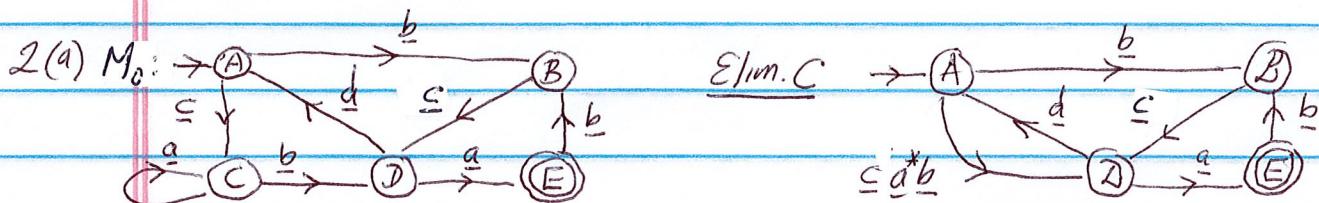
$$(a) L_1 = \{a^k b^n : k \pmod 3 > (2n-1) \pmod 3\} \quad (b) L_2 = \{b^k c^n : k < n^2 + 3\}.$$

[If you say that it is regular, you must find a regular expression for it; if you say it is non-regular, you must give a complete proof.]



$$D \rightarrow aE, E \rightarrow \lambda,$$

(b)  $G: \rightarrow A, A \rightarrow bB, A \rightarrow cC, C \rightarrow aC, C \rightarrow bD, B \rightarrow cD, D \rightarrow aA, E \rightarrow bB.$



$$L(M) = R_1^* R_2 (R_4 + R_3 R_1^* R_2)^*$$

$$= (\underline{bcd} + \underline{ca^*bd})^* \cdot (\underline{bca} + \underline{ca^*ba}) \cdot (\underline{bca} + \underline{bcd} \cdot (\underline{bcd} + \underline{ca^*bd})^* \cdot (\underline{bca} + \underline{ca^*ba}))^*$$

(b)  $\beta(n) =$  maximum number of 1's that a TM  $M$  in  $\mathcal{M}_n$  can produce when started on the blank tape.  $\mathcal{M}_n =$  set of all TMs with  $n$  states and tape alphabet  $\{1, \lambda\}$  which will halt when started on the blank tape.

3(a) The initial functions are (i) the constant 0, (ii) the zero function,  $z(x) \equiv 0$ , (iii) the successor function  $s(x) = x+1$ ; and the projective functions

$$I_{k,n}(x_1, \dots, x_n) = x_k \text{ if } 1 \leq k \leq n, \text{ and } \lambda \text{ if } k=0.$$

Primitive recursion is the operation that produces a function  $f: \mathbb{N}^{n+1} \rightarrow \mathbb{N}$  from the functions  $g: \mathbb{N}^n \rightarrow \mathbb{N}$   $\quad f(\underline{x}, 0) = g(\underline{x})$  where  $\underline{x} = (x_1, \dots, x_n)$  and  $h: \mathbb{N}^{n+2} \rightarrow \mathbb{N}$  by putting  $\quad f(\underline{x}, s(y)) = h(\underline{x}, y, f(\underline{x}, y)).$

(b) We will find primitive recursive functions  $g$  &  $h$  such that  $f = \text{prec}(g, h)$ . We have  $f(\underline{x}, 0) = 2x + 3(0) + 1 = 2x + 1 \leftarrow g(\underline{x})$  and  $f(\underline{x}, s(y)) = 2x + 3(x+1) + 1 = (2x + 3y + 1) + 3 = f(\underline{x}, y) + 3 \leftarrow h(\underline{x}, y, f(\underline{x}, y)).$

3(b)  $\therefore g(x) = 2x+1$ , so  $g(y) = 2y+1$  and  $h = \text{so so so } I_{3,3}$ . Now

$g(0) = 1$  and  $g(s(y)) = 2(y+1)+1 = (2y+1)+2 = g(y)+2$ . Hence

$g = \text{prec}(\text{so0}, \text{so so so } I_{2,2})$ .  $\therefore f = \text{prec}(\text{prec}(\text{so0}, \text{so so so } I_{2,2}), \text{so so so so } I_{3,3})$ .

4(a) A  $\mu$ -recursive function is any partial function  $f: \mathbb{N}^k \rightarrow \mathbb{N}$  that can be obtained from the initial functions by a finite no. of applications of cartesian products, compositions, primitive recursions, and minimizations on total functions.

(b) We will find a total function  $g: \mathbb{N}^2 \rightarrow \mathbb{N}$  s.t.  $f = \mu[g, 0]$ .

Let  $g(x, y) = (2x+1) \dot{-} y^2$ . Then  $(\mu y)[(2x+1) \dot{-} y^2 = 0] = \lceil (2x+1)^{1/2} \rceil = f(x)$ .

So  $f = \mu [\text{MONUS} \circ (\text{so ADD} \circ [I_{1,2} \hat{\wedge} I_{1,2}] \hat{\wedge} \text{MULT} \circ [I_{2,2} \hat{\wedge} I_{2,2}])], 0$ .

5(a)  $R \subseteq \mathbb{N}^k$  is Turing-decidable if we can find a TM M such that on the input  $\underline{x}$ ; M will halt in an accepting state if  $\underline{x} \in R$ , and M will halt in a non-accepting state if  $\underline{x} \notin R$ . Here  $\underline{x} = \langle x_1, \dots, x_k \rangle$

(b) (i)  $\langle A, 1 \rangle \vdash \langle A, a \underline{\underline{u}} \rangle \vdash \langle B, a \rangle \vdash \langle C, 1 \underline{\underline{u}} \rangle \vdash \langle D, 11 \underline{\underline{u}} \rangle \vdash \langle B, 111 \rangle \vdash \langle B, 111 \rangle \vdash \langle B, \underline{111} \rangle \vdash \langle E, \underline{\underline{1111}} \rangle \vdash \langle F, \underline{1111} \rangle$  halts

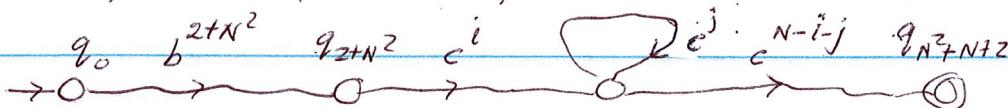
(ii)  $\langle A, \underline{\underline{u}} \rangle \vdash \langle B, \underline{\underline{u}} \rangle \vdash \langle E, \underline{\underline{111}} \rangle \vdash \langle F, 1 \rangle$  halts

(c)  $f(n) = 3n+1$  because  $f(0) = 1$ ,  $f(1) = 4$ , and you can check  $f(2) = 7$ .

6(a)  $n \equiv 0 \pmod{3} \Rightarrow k \pmod{3} > 2(0) - 1 \equiv 2 \pmod{3} \Rightarrow$  no value of  $k$   
 $n \equiv 1 \pmod{3} \Rightarrow k \pmod{3} > 2(1) - 1 \equiv 1 \pmod{3} \Rightarrow k \equiv 1 \pmod{3}$   
 $n \equiv 2 \pmod{3} \Rightarrow k \pmod{3} > 2(2) - 1 \equiv 0 \pmod{3} \Rightarrow k \equiv 2 \pmod{3}$

'.'  $\underline{aa}(aaa)^* \cdot \underline{b}(bbb)^* + (a+a\underline{a})(aaa)^* \cdot \underline{bb}(bbb)^*$  is a reg. expr. for  $L_1$ .

(b) Suppose  $L_2$  was regular. Then we can find a  $\lambda$ -free NFA M (with  $N$  states) such that  $\mathcal{L}(M) = L_2$ . Since  $2+N^2 < 3+(N)^2$ ,  $b^{2+N^2}c^N \in L_2 = \mathcal{L}(M)$ . Since it takes  $N+1$  states to process  $c^N$ , the acceptance track of  $b^{2+N^2}c^N$  must have a loop as shown.



Now if we skip the loop, we see that M will accept  $b^{2+N^2}c^{N-j}$ . But  $2+N^2 \neq 3+(N-j)^2$  because  $N, j \geq 1$ . So this contradicts the fact that  $\mathcal{L}(M) = L_2$ . Hence  $L_2$  is not a regular language.