

Answer all 6 questions. No calculators, formula sheets or cell-phones are allowed. Provide all reasoning and show all working. An unjustified answer will receive little or no credit. Begin each of the 6 questions on 6 separate pages.

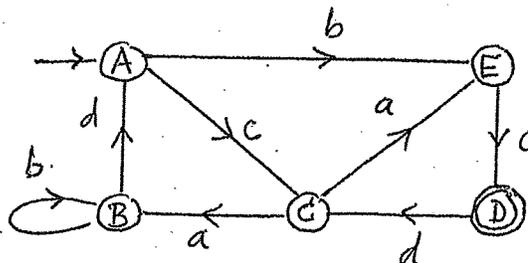
- (15) 1. (a) Find an NFA,  $M$ , which is equivalent to the RLG  $G$  given below.

$G: \rightarrow B, B \rightarrow 10, B \rightarrow 0B, B \rightarrow 0C, C \rightarrow 10,$   
 $C \rightarrow D, C \rightarrow 0E, E \rightarrow 1D, E \rightarrow \lambda.$

- (b) Find an RLG,  $G$ , which is equivalent to the NFA in Problem 2 below.

- (15) 2. (a) Find a regular expression for the language accepted by the NFA shown on the right.

- (b) Define what is the Busy Beaver function.



- (15) 3. (a) Define what it is a primitive recursive function.

- (b) Show that  $f(x,y) = 4x+3y+1$  is a primitive recursive function by finding primitive recursive functions  $g$  and  $h$  such that  $f = \text{prec}[g,h]$ .

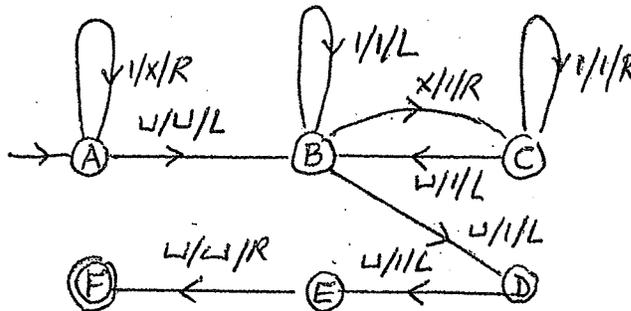
- (20) 4. (a) Define what it means for the function  $f: \mathbb{N}^n \rightarrow \mathbb{N}$  to be obtained from the total function  $g: \mathbb{N}^{n+1} \rightarrow \mathbb{N}$  by minimization.

- (b) Let  $f(x) = \lceil (x^{1/2}) \rceil$  and  $h(x)=0$  (if  $x$  is a perfect square) &  $h(x)=1$  (otherwise). Show that  $f$  and  $h$  are  $\mu$ -recursive functions.

[You may use the fact that PRED, MONUS, ADD, MULT & SIGN are primitive recursive if needed in #4, but you are not allowed to do so in #3.]

- (15) 5. (a) Define what is a Turing computable function  $f: D \rightarrow \mathbb{N}$ , where  $D \subseteq \mathbb{N}$ .

- (b) Show what happens at each step if (i)  $\lambda$  and (ii)  $11$  are the inputs for the TM,  $M$ , shown on the right.



- (20) 6. Determine which of the following languages are regular and which are not.

(a)  $L_1 = \{a^k.b^n : k \pmod{3} < 1+n^2 \pmod{3}\}$  (b)  $L_2 = \{b^k.a^n : k < 1+n^2\}$ .

[If you say that it is regular, you must find a regular expression; if you say it is non-regular, you must give a complete proof.]



4(a)  $f(x) = \begin{cases} \text{smallest value of } y \text{ such that } g(x, y) = 0; \\ \text{undefined, if } g(x, y) > 0 \text{ for all } y \in \mathbb{N}. \end{cases}$   $x = \langle x_1, \dots, x_n \rangle$

(b) Let  $g(x, y) = x \div y^2$ . Then  $(\mu y)[g(x, y) = 0] = (\mu y)[x \div y^2 = 0] = \lceil x^{1/2} \rceil = f(x)$ . So  $f = \mu[g, 0] = \mu[\text{MONUS} \circ \{I_{1,2} \wedge \text{MULT} \circ (I_{2,2} \wedge I_{2,2})\}, 0]$ .  
 $\therefore f$  is  $\mu$ -recursive

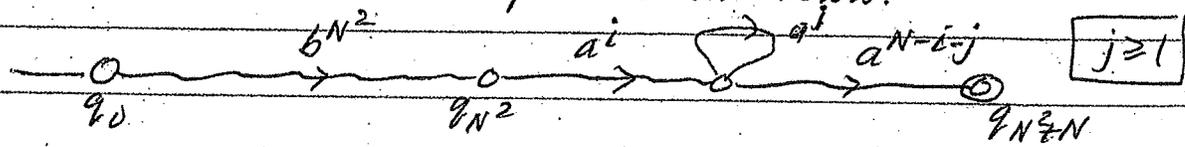
(c)  $h(x) = \text{SIGN}\{\lceil x^{1/2} \rceil^2 \div x\} = \text{SIGN}\{f(x)^2 \div x\}$ . So  $h = \text{SIGN} \circ \text{MONUS}\{\text{MULT} \circ (f \wedge f) \wedge I_{1,2}\}$ .  $\therefore h$  is  $\mu$ -recursive.

5(a)  $f: D \rightarrow \mathbb{N}$  is Turing computable if we can find a TM  $M$  such that  $w \in D \iff \langle q_0, w \rangle \vdash \langle q_f, f(w) \rangle$  is a halted computation with  $q_f \in A(M)$ .

(b) (i)  $\langle A, \_ \rangle \vdash \langle B, \_ \_ \rangle \vdash \langle D, \_ \_ \_ \rangle \vdash E \langle \_ \_ \_ \rangle \vdash \langle F, \_ \_ \rangle$ .  
 (ii)  $\langle A, \_ \_ \rangle \vdash \langle A, x \_ \rangle \vdash \langle A, x x \_ \rangle \vdash \langle B, x x \_ \rangle \vdash \langle C, x \_ \_ \rangle$   
 $\vdash \langle B, x \_ \_ \rangle \vdash \langle B, x \_ \_ \rangle \vdash \langle C, \_ \_ \_ \rangle \vdash \langle C, \_ \_ \_ \rangle \vdash \langle C, \_ \_ \_ \rangle \vdash \langle B, \_ \_ \_ \rangle$   
 $\vdash \langle B, \_ \_ \_ \rangle \vdash \langle B, \_ \_ \_ \rangle \vdash \langle B, \_ \_ \_ \rangle \vdash \langle D, \_ \_ \_ \rangle \vdash \langle E, \_ \_ \_ \rangle \vdash \langle F, \_ \_ \_ \rangle$

6(a)  $n \equiv 0 \pmod{3} \Rightarrow k \pmod{3} < 1 + 0^2 \pmod{3} \equiv 1 \pmod{3} \Rightarrow k \equiv 0 \pmod{3}$   
 $n \equiv 1 \pmod{3} \Rightarrow k \pmod{3} < 1 + 1^2 \pmod{3} \equiv 2 \pmod{3} \Rightarrow k \equiv 0 \text{ or } 1 \pmod{3}$   
 $n \equiv 2 \pmod{3} \Rightarrow k \pmod{3} < 1 + 2^2 \pmod{3} \equiv 2 \pmod{3} \Rightarrow k \equiv 0 \text{ or } 1 \pmod{3}$   
 So  $(aaa)^*(bbb)^* + (\lambda + a)(aaa)^* \cdot b \cdot (bbb)^2 + (\lambda + a)(aaa)^* \cdot bb(bbb)^*$  is reg. expr. for  $L_1$ . So  $L_1$  is regular.

(b) Suppose  $L_2 = \{b^k a^n : k < 1 + n^2\}$  was regular. Then we can find a  $\lambda$ -free NFA  $M$  with  $N$  states such that  $L(M) = L_2$ . Now  $b^{N^2} a^N \in L_2$ , because  $N^2 < 1 + (N)^2$ . So  $M$  will accept  $b^{N^2} a^N$ . Since it takes  $N+1$  states to process the  $a^N$ , any acceptance track of  $b^{N^2} a^N$  must contain a loop as shown below.



Now if we skip the loop, we will see that  $M$  accepts the string  $b^{N^2} a^{N-j}$ . But  $N^2 \not< 1 + (N-j)^2$ , so this contradicts the fact that  $L(M) = L_2$ . Hence  $L_2$  is a non-regular language.