

Answer all 6 questions. No calculators, formula sheets or cell-phones are allowed.
 Provide all reasoning and show all working. An unjustified answer will receive little or no credit. Begin each of the 6 questions on 6 separate pages.

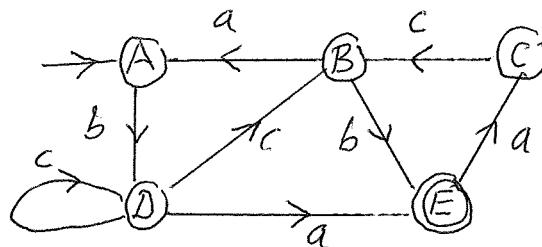
- (17) 1. (a) Find an NFA, M , which is equivalent to the RLG G given below.

$$G: \quad \rightarrow A, \quad A \rightarrow 01, \quad A \rightarrow 1B, \quad B \rightarrow 0C, \quad C \rightarrow 10, \\ C \rightarrow \lambda, \quad C \rightarrow D, \quad D \rightarrow 0E, \quad E \rightarrow 10E, \quad E \rightarrow \lambda.$$

- (b) Find an RLG, G , which is equivalent to the NFA in Problem 2 below.

- (15) 2. (a) Find a regular expression for the language accepted by the NFA shown on the right.

- (b) Write down what is the Halting Problem asks.



- (15) 3. (a) Define what is the operation called primitive recursion.

- (b) Show that $f(x,y) = 2x+4y+3$ is a primitive recursive function by finding primitive recursive functions.

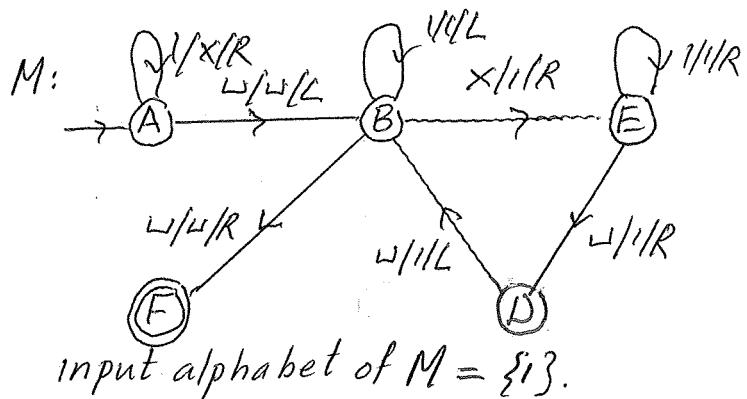
- (15) 4. (a) Define what it means for the function f (of n variables) to be obtained from the total function g (of $n+1$ variables) by minimization.

- (b) Let $f(x) = \text{Ceiling function of } [(3x+2)^{1/2}]$. Show that f is a μ -recursive function.
 [You may use the fact that PRED, MONUS, ADD, MULT & SIGN are primitive recursive if needed in #4, but you are not allowed to do so in #3.]

- (18) 5. (a) Define what is a Turing semi-decidable relation R on \mathbb{N} .

- (b) Show what happens at each step if (i) λ and (ii) 1 are the inputs for the TM, M , shown on the right.

- (c) What is the function computed by M ? (in monadic notation)



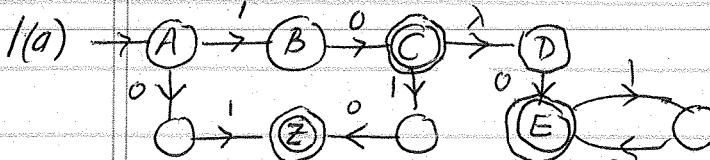
- (20) 6. Determine which of the following languages are regular and which are not.

$$(a) L_1 = \{a^k b^n : k \pmod{3} < (n^2 - 2) \pmod{3}\} \quad (b) L_2 = \{b^k c^n : k > 1 + n^2\}.$$

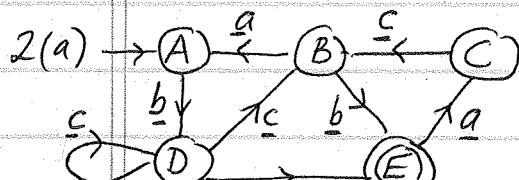
[If you say that it is regular, you must find a regular expression; if you say it is non-regular, you must give a complete proof.]

Solutions to Test #2

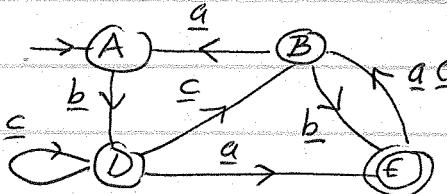
Spring 2017



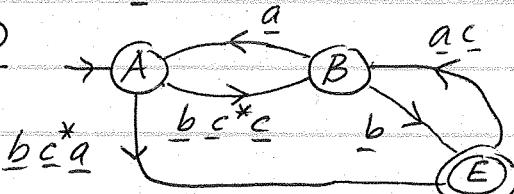
(b) $\rightarrow A, A \rightarrow bD, D \rightarrow cD, D \rightarrow cB$
 $D \rightarrow aE, E \rightarrow \lambda, E \rightarrow aC, C \rightarrow cB$
 $B \rightarrow bE, B \rightarrow aA.$



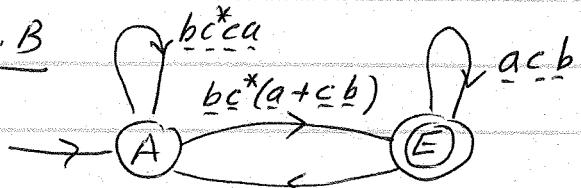
Elm. C



Elm. D



Elm. B



$$L(M) = (bc^*c^q)^* \cdot bc^*(a+cb) \cdot [acb + acb (bc^*c^q)^* \cdot bc^*(a+cb)]^*$$

3 (a) Primitive recursion is the operation that produces a function $f: \mathbb{N}^{n+1} \rightarrow \mathbb{N}$ from the functions $g: \mathbb{N}^n \rightarrow \mathbb{N}$ & $h: \mathbb{N}^{n+2} \rightarrow \mathbb{N}$ by setting $f(x, 0) = g(x)$ and $f(x, s(y)) = h(x, y, f(x, y))$.

(b) $f(x, y) = 2x + 4y + 3$. So $f(x, 0) = 2x + 3$. Here $\underline{x} = (x_1, \dots, x_n)$

Now if we want $f = \text{prec}(g, h)$ then we must have $g(x) = 2x + 3$, and $f(x, s(y)) = 2x + 4(y+1) + 3$

$$= (2x + 4y + 3) + 4 = f(x, y) + 4, \text{ so } h(x, y, f(x, y)) = f(x, y) + 4.$$

$\therefore h = \text{so so so so } I_{3,3}$. Also $g(0) = 2(0) + 3 = 3$ and

$$g(s(y)) = 2(y+1) + 3 = (2y+3) + 2 = g(y) + 2. \text{ Hence } g = \text{prec}(g, h)$$

where $g_1 = \text{so so so } 0$ & $h_1 = \text{so so so } I_{2,2}$. Thus $g = \text{prec}(\text{so so so } 0, \text{so so so } I_{2,2})$

$\therefore f = \text{prec}(g, h) = \text{prec}(\text{prec}(\text{so so so } 0, \text{so so so } I_{2,2}), \text{so so so so } I_{3,3})$

4 (a) Minimization is the operation that produces a partial function $f: \mathbb{N}^n \rightarrow \mathbb{N}$ from a total function $g: \mathbb{N}^{n+1} \rightarrow \mathbb{N}$ by setting $f(x) = \begin{cases} \text{the smallest } y \text{ such that } g(x, y) = 0, & x = (x_1, \dots, x_n) \\ \text{undefined, if there is no value of } y \text{ s.t. } g(x, y) = 0. \end{cases}$

4(b) Let $g(x, y) = (3x+2) - y^2$. Then $(\mu y)[g(x, y)=0] = \lceil (3x+2)^{1/2} \rceil$
So $f = \mu[g, 0] = \mu[\text{MONUS}_0[S_0 S_0 \text{MULT}_0(S_0 S_0 S_0 Z_0 I_{1,2}) \wedge I_{1,2}] \wedge$
 $\text{I. } f \text{ is } \mu\text{-recursive. }$ $\text{MULT}_0[I_{2,2} \wedge I_{2,2}]], 0]$

5(a) A relation $R \subseteq N \times N$ is Turing semi-decidable if we can find a TM M such that M halts on $\langle m, n \rangle$ in an accepting state if $\langle m, n \rangle \in R$ and M halts on $\langle m, n \rangle$ in a non-acc. state or M fails to halt, $f(m, n) \notin R$.

$$(b)(i) \langle A, \underline{\cup} \underline{\cup} \underline{\cup} \rangle \vdash \langle B, \underline{\cup} \underline{\cup} \underline{\cup} \rangle \vdash \langle F, \underline{\cup} \underline{\cup} \underline{\cup} \rangle$$

$$(ii) \langle A, \underline{1} \rangle \vdash \langle A, \underline{x} \underline{\cup} \rangle \vdash \langle B, \underline{x} \underline{\cup} \rangle \vdash \langle E, \underline{1} \underline{\cup} \rangle \vdash \langle D, \underline{1} \underline{\cup} \underline{\cup} \rangle \\ \vdash \langle B, \underline{1} \underline{1} \underline{1} \rangle \vdash \langle B, \underline{1} \underline{1} \underline{1} \rangle \vdash \langle B, \underline{\cup} \underline{1} \underline{1} \rangle \vdash \langle F, \underline{1} \underline{1} \underline{1} \rangle$$

$$(c) f(0) = 0, f(1) = 3, f(2) = 6 \text{ (check this). So } f(n) = 3n \text{ in monadic notation.}$$

$$6(a) n \equiv 0 \pmod{3} \Rightarrow k \pmod{3} < (0^2 - 2) \pmod{3} = 1 \pmod{3} \Rightarrow k = 0 \pmod{3}$$

$$n \equiv 1 \pmod{3} \Rightarrow k \pmod{3} < (1^2 - 2) \pmod{3} = 2 \pmod{3} \Rightarrow k = 0 \text{ or } 1 \pmod{3}$$

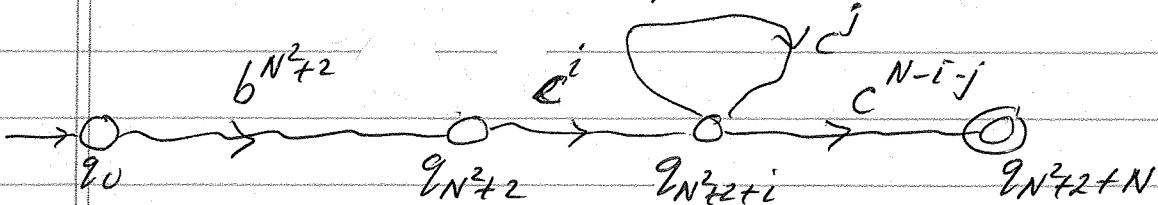
$$n \equiv 2 \pmod{3} \Rightarrow k \pmod{3} < (2^2 - 2) \pmod{3} = 2 \pmod{3} \Rightarrow k = 0 \text{ or } 1 \pmod{3}$$

So $(a_a_a)^*(b_b_b)^* + (\underline{a+a})(\underline{a_a_a})^* b(\underline{b_b_b})^* + (\underline{a+a})(\underline{a_a_a})^* b\underline{b}(\underline{b_b_b})^*$ is a reg. expression for L_1 . Hence L_1 is a regular language.

(b) Suppose $L_2 = \{b^{k^2} c^n : k > 1+n^2\}$ was regular. Then we can find a λ -free NFA M with N states such that $L(M) = L_2$.

Now $b^{N^2+2} c^N \in L_2$ because if we put $k = N^2+2$ & $n=N$, then $k > 1+n^2$.

Since it takes $N+1$ states to process the c^N , any acceptance track of $b^{N^2+2} c^N$ must contain a loop as shown below, with $j \geq 1$.



Now if we ride the loop twice, we will see that M accepts the string $b^{N^2+2} c^i c^j c^{N-i-j} = b^{N^2+2} c^{N+j}$. But $N^2+2 \neq 1+(N+j)^2 = 1+N^2+2jN+j^2$ because $j \geq 1$. So this contradicts the fact that $L(M) = L_2$. Hence L_2 is a non-regular language.

END