

MAD 3512 - THEORY OF ALGORITHMS

TEST #2 - Sp 2020

FLORIDA INT'L UNIV.

TIME: 75 min.

Answer all 6 questions. No calculators, formula sheets or cell-phones are allowed. Provide all reasoning and show all working. An unjustified answer will receive little or no credit. Begin each of the 6 questions on 6 separate pages.

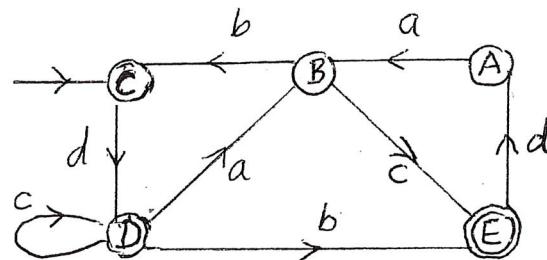
- (16) 1. (a) Find an NFA,  $M$ , which is equivalent to the RLG  $G$  given below.

$$G: \rightarrow B, \quad B \rightarrow 01B, \quad B \rightarrow 1C, \quad C \rightarrow 11, \quad C \rightarrow \lambda, \quad C \rightarrow 1D, \\ C \rightarrow E, \quad D \rightarrow 1B, \quad D \rightarrow \lambda, \quad E \rightarrow 0C, \quad E \rightarrow 01.$$

- (b) Find an RLG,  $G$ , which is equivalent to the NFA in Problem 2(a) below.

- (16) 2. (a) Find a regular expression for the language accepted by the NFA  $M$  shown on the right.

- (b) Define what is the busy-beaver function,  $\beta(n)$ .



- (16) 3. (a) Define the initial functions and the operation called primitive recursion.

- (b) Show that  $f(x,y) = 2x+4y+3$  is a primitive recursive function by finding primitive recursive functions  $g$  and  $h$  such that  $f = \text{prec}(g,h)$ .

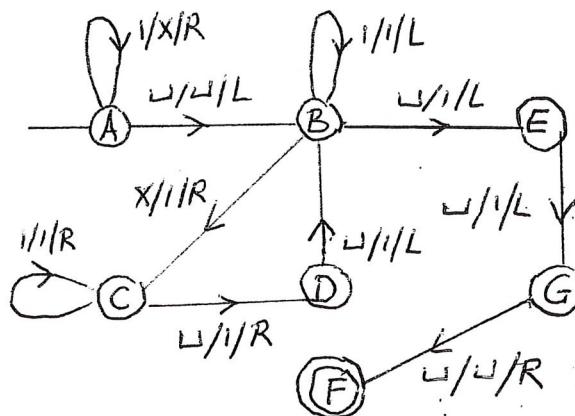
- (16) 4.(a) What is the difference between a total function & a  $\mu$ -recursive function on  $N$ .

- (b) Let  $f(x) = \text{Ceiling function of } [(x^2+3)^{1/2}]$ . Show that  $f$  is a  $\mu$ -recursive function.  
[You may use the fact that PRED, MONUS, ADD, MULT & SIGN are primitive recursive if needed in #4, but you are not allowed to do so in Question #3.]

- (18) 5.(a) What is the difference between a Turing-decidable language and a Turing semi-decidable language.

- (b) Show what happens at each step if (i) 1 and (ii)  $\lambda$  are the inputs for the TM,  $M$ , shown on the right.

- (c) What is the function computed by  $M$  in monadic (base 1) notation?



- (18) 6. Determine which of the following languages are regular and which are not.

- (a)  $L_1 = \{a^k b^n : k \pmod 3 < (n^2 - 2) \pmod 3\}$  (b)  $L_2 = \{b^k c^n : k > n^2 + 4\}$ .

[If you say that it is regular, you must find a regular expression for it; if you say it is non-regular, you must give a complete proof.]

Solutions to Test #2

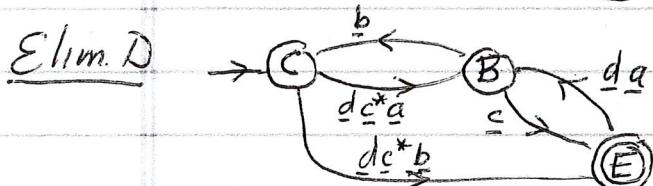
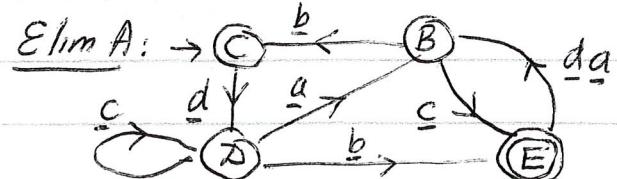
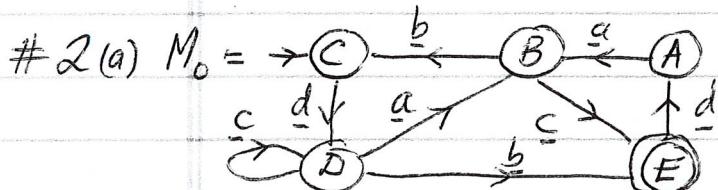
Spring 2020

#1(a)



$$E \rightarrow dA, A \rightarrow aB.$$

(b)  $\rightarrow C, C \rightarrow dD, D \rightarrow cD, D \rightarrow aB, D \rightarrow bE, B \rightarrow bC, B \rightarrow cE, E \rightarrow \lambda, \lambda$



$$\begin{aligned} L(M) &= R_1^* R_2 (R_4 + R_3 R_1^* R_2)^* \\ &= (\underline{dc^*ab})^* (\underline{dc^*ac} + \underline{dc^*b}) [\underline{dac} + (\underline{dab}) \cdot (\underline{dc^*ab})^*, (\underline{dc^*ac} + \underline{dc^*b})]^* \end{aligned}$$

2(b)  $\beta(n) =$  maximum number of 1's that a TM in  $\mathcal{H}_n$  can produce when started on the blank tape. Here  $\mathcal{H}_n$  is the set of all TMs with  $n$  states and tape alphabet  $\{1, \leftarrow\}$  which halts when started on the blank tape.

#3. The initial functions are: (i) the constant 0, (ii) the zero function  $I(x) \equiv 0$ , (iii) the successor function  $S(x) = x+1$ , and (iv) the projective functions  $I_{k,n}(x_1, \dots, x_n) = x_k$  if  $1 \leq k \leq n$ , and  $\lambda$  if  $k=0$ .

Primitive recursion is the operation that produces a function  $f: N^{n+1} \rightarrow N$  from the functions  $g: N^n \rightarrow N$  &  $h: N^{n+2} \rightarrow N$  by putting  $f(\underline{x}, 0) = g(\underline{x})$  and  $f(\underline{x}, s(y)) = h(\underline{x}, y, f(\underline{x}, y))$ . Here  $\underline{x} = \langle x_1, \dots, x_n \rangle$

3(b) We will find primitive recursive functions  $g$  &  $h$  such that  $f = \text{prec}(g, h)$ . We have  $f(\underline{x}, 0) = 2x + 4(0) + 3 = 2x + 3$ ,  $\leftarrow g(\underline{x})$  and  $f(\underline{x}, s(y)) = 2x + 4(y+1) + 3 = 2x + 4y + 3 + 4 = f(\underline{x}, y) + 4$ ,  $\leftarrow h(\underline{x}, y, f(\underline{x}, y))$

$\therefore h = s_0 s_0 s_0 s_0 I_{3,3}$ . Also we will express  $g$  as  $\text{prec}(g_1, h_1)$  where  $g_1: \mathbb{N}^0 \rightarrow \mathbb{N}$  &  $h_1: \mathbb{N}^2 \rightarrow \mathbb{N}$ . Now  $g(x) = 2x+3$ , so  $g(y) = 2y+3$   
 $\therefore g(0) = 3$  and  $g(s(y)) = g(y+1) = 2(y+1)+3 = 2y+3+2 = g(y)+2$   
 $\therefore g_1 = 3 = s_0 s_0 s_0 0$  &  $h_1 = s_0 s_0 I_{2,2}$ .  $\therefore g = \text{prec}(s_0 s_0 s_0 0, s_0 s_0 I_{2,2})$   
 Thus  $f = \text{prec}(\text{prec}(s_0 s_0 s_0 0, s_0 s_0 I_{2,2}), s_0 s_0 s_0 s_0 I_{3,3})$ .

#4(a) A function  $f: \mathbb{N}^n \rightarrow \mathbb{N}$  is a total function if  $f$  is defined for each  $x \in \mathbb{N}^n$ . A  $\mu$ -recursive function  $f: \mathbb{N}^n \rightarrow \mathbb{N}$  is any partial function which can be obtained from the initial functions by a finite number of applications of compositions, cartesian products, primitive recursions, and minimization on total functions. The difference is that total functions have no restriction in the way they can be defined while  $\mu$ -recursive functions are restricted in the way they are defined but they can be undefined at some  $x$ .

4(b) We will find a total, primitive-recursive, function  $g: \mathbb{N}^2 \rightarrow \mathbb{N}$  such that  $f = \mu[g, 0]$ . Let  $g(x, y) = x^2 + 3 - y^2$ .  
 Then  $(\forall y)[(x^2 + 3 - y^2 = 0)] = [(x^2 + 3)^{1/2}] = f(x)$ . So  $f = \mu[g, 0]$   
 $= \mu[\text{MONUS} \circ (s_0 s_0 s_0 \text{MULT} \circ [I_{1,2} \wedge I_{1,2}] \wedge \text{MULT}_0 [I_{2,2} \wedge I_{2,2}], 0)]$ .

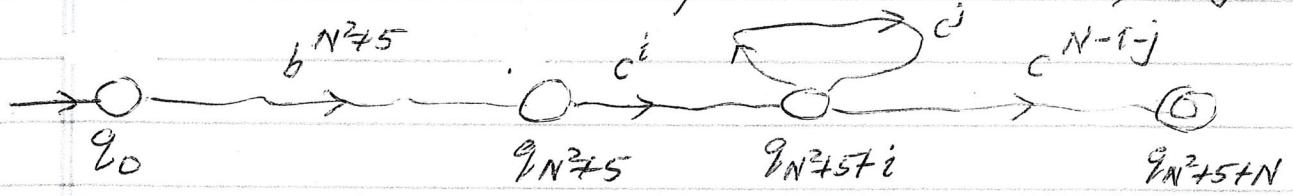
#5. A language  $L$  is Turing-decidable if we can find a TM  $M_1$  such that  $M_1$  halts on  $w$  in an accepting state, if  $w \in L$  and  $M_1$  halts on  $w$  in a non-accepting, if  $w \notin L$  — while  $L$  is Turing semi-decidable if we can find a TM  $M_2$  such that  $M_2$  halts in an accepting, if  $w \in L$  and  $M_2$  halts on  $w$  in a non-accepting state or fails to halt, if  $w \notin L$ . The difference is that when  $w \notin L$ ,  $M_2$  does not have to halt and say NO —  $M_2$  can fail to halt.  $M_2$  just has to say YES if  $w \in L$ . Of course, if  $w \notin L$ ,  $M_2$  cannot say YES.

5(b)  $\langle A, ! \rangle \vdash \langle A, x \sqsubseteq \rangle \vdash \langle B, \leq \rangle \vdash \langle C, 1 \sqsubseteq \rangle \vdash \langle D, 11 \sqsubseteq \rangle$   
 $\vdash \langle B, 111 \rangle \vdash \langle B, 111 \rangle \vdash \langle B, \sqsubseteq 111 \rangle \vdash \langle E, \sqsubseteq 1111 \rangle \vdash \langle G, \sqsubseteq 11111 \rangle$   
 $\vdash \langle F, \sqsubseteq 11111 \rangle \text{ halts}$   
 $\langle A, \sqsubseteq \rangle \vdash \langle B, \sqsubseteq \sqsubseteq \rangle \vdash \langle E, \sqsubseteq 1 \sqsubseteq \rangle \vdash \langle G, \sqsubseteq 11 \sqsubseteq \rangle \vdash \langle F, \sqsubseteq 1 \rangle \text{ halts}$

5(c)  $f(0)=2$ ,  $f(1)=5$ , and you can check that  $f(2)=8$ .  $f(x)=3x+2$ .

#6(a)  $n \equiv 0 \Rightarrow k \pmod{3} < 0^2 - 2 \equiv 1 \pmod{3} \Rightarrow k \equiv 0 \pmod{3}$   
 $n \equiv 1 \Rightarrow k \pmod{3} < 1^2 - 2 \equiv 2 \pmod{3} \Rightarrow k \equiv 0 \text{ or } 1 \pmod{3}$   
 $n \equiv 2 \Rightarrow k \pmod{3} < 2^2 - 2 \equiv 2 \pmod{3} \Rightarrow k \equiv 0 \text{ or } 1 \pmod{3}$   
 $\therefore (\underline{aaa})^*(\underline{bbb})^* + (\underline{a+a})(\underline{aa})^* b (\underline{bbb})^* + (\underline{a+a})(\underline{aaa})^* b b (\underline{bbb})^*$  is  
a regular expression which describes  $L_1$ . So  $L_1$  is regular.

6(b) Suppose  $L_2$  was regular. Then we can find a  $\lambda$ -free  
NFA  $M$  with  $N$  states such that  $\mathcal{L}(M) = L_2$ . Since  
 $N^2 + 5 > N^2 + 4$ , if we take  $k = N^2 + 5$  and  $n = N$ , then we  
would have  $b^k c^n \in L_2$  because  $k > (n)^2 + 4$ . Since it  
takes  $N+1$  states to process  $c^N = c^n$ , the acceptance track  
of  $b^{N^2+5} c^N$  must have a loop as shown below, with  $j \geq 1$ .



Now if we ride this loop twice, we will see that  
 $M$  accepts  $b^{N^2+5} c^i c^j c^i c^{N-i-j} = b^{N^2+5} c^{N+j}$ . But  
 $N^2 + 5 \not> (N+j)^2 + 4$  because  $(N+j)^2 = N^2 + 2Nj + j^2 + 4$   
 $\geq N^2 + 7$ , since  $N, j \geq 1$ .

So this contradicts the fact that  $\mathcal{L}(M) = L_2$ . Hence  
 $L_2$  is not a regular language. END