

$$\delta(q_0, a) = (q_1, a, R)$$

$$Q = \{q_0, q_1\}, \quad F = \{q_1\}, \quad \Sigma = \{a, b\}, \quad \Gamma = \{a, b, \square\}$$

3 aba :

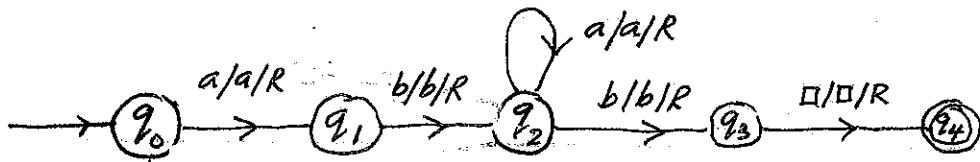
$$q_0 \xrightarrow{} aba \xrightarrow{} q_1 ba \xrightarrow{} q_2 xy a \xrightarrow{} q_0 ya \xrightarrow{} xy q_3 a$$

4 No.

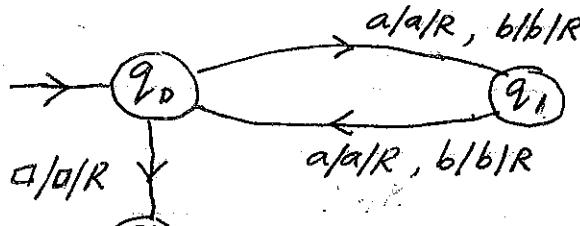
5 $L(a b^* + b b^* a (a+b)^*)$

6. The Turing Machine halts in a non-accepting state

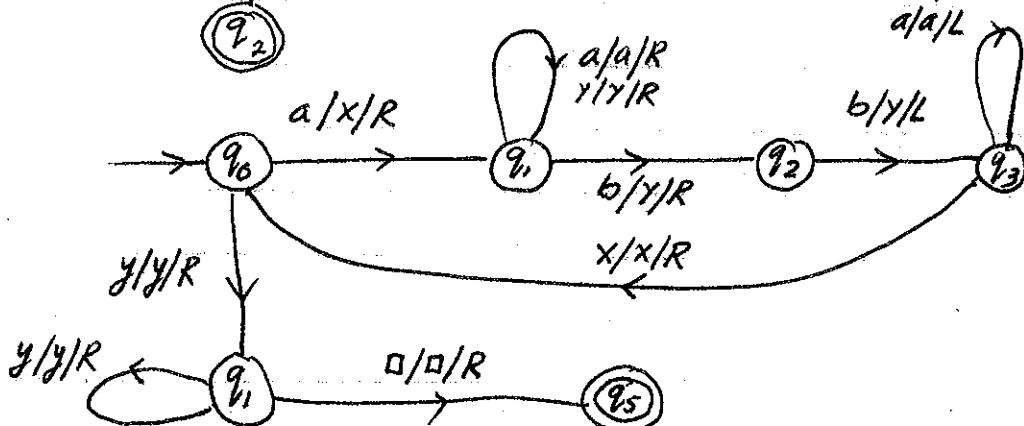
7 (a)



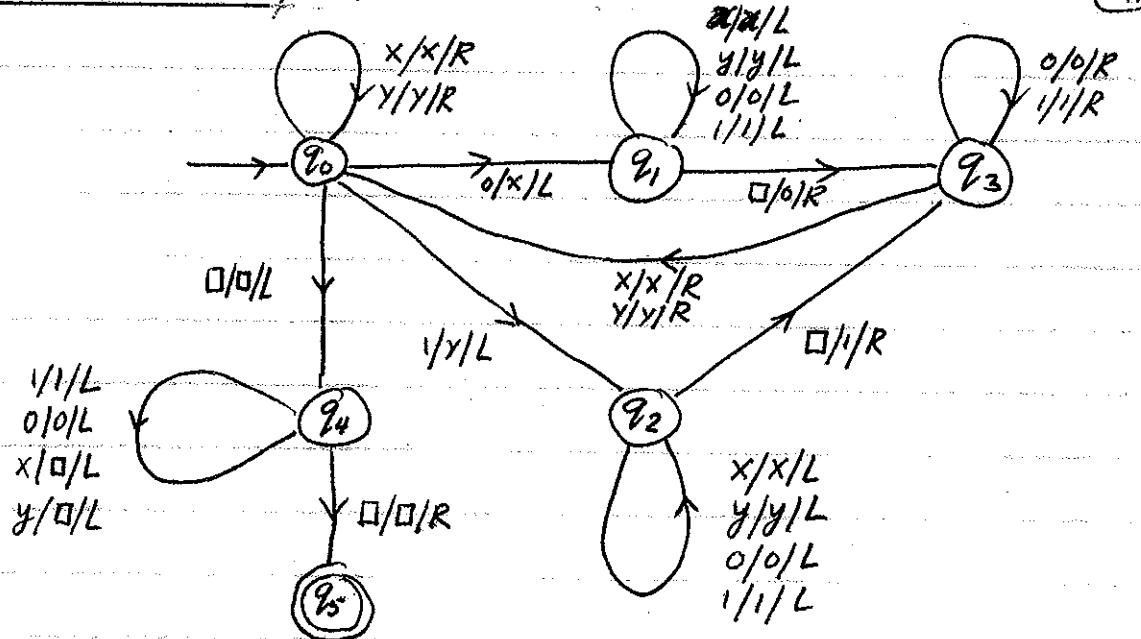
(b)



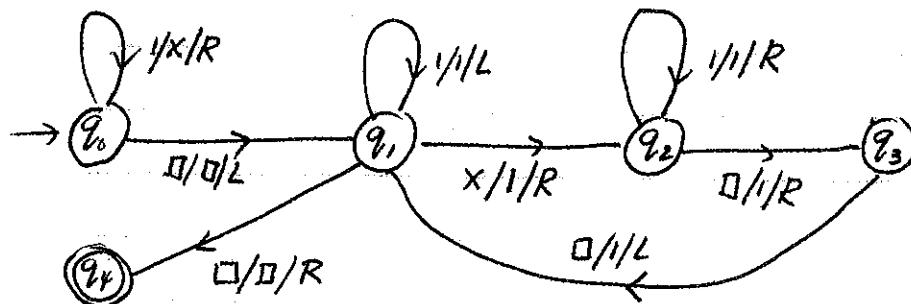
(c)



#9.



#11(a) Modify the machine in Example 9.10 to write two 1's for each x :



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- 12 (a) If L_1 is recursive and L_2 is r.e., then $L_2 - L_1$ is necessarily r.e. because \bar{L}_1 is r.e.,
So $L_2 - L_1 = L_2 \cap \bar{L}_1$ = intersection of two r.e. sets = r.e
- (b) If L_1 is recursive and L_2 is r.e., then $L_1 - L_2$ is not necessarily r.e. because if we take $L_1 = \Sigma^*$ and $L_2 =$ a non-recursive r.e. subset of Σ^* , then $L_1 - L_2 =$ a non-r.e. subset of Σ^* .

#2 The set of all r.e. languages is countable since every r.e. language is associated with a TM and the number of different TM's is countable. If there were countably many non-r.e. languages as well, then there would be countably many languages altogether. But the set of all languages on Σ is uncountable because $\mathcal{L}(\Sigma) = \mathcal{P}(\Sigma^*) \approx \mathcal{P}(N)$ which is uncountable. So there are uncountably many non-r.e. languages.

#5 Suppose \bar{L} is recursive. Then \bar{L} will also be recursive by the closure theorem for recursive languages. But $\bar{\bar{L}} = L$. So L is recursive. Hence L is r.e. But we were told that L was not r.e. So if L is not r.e., then \bar{L} cannot be recursive.

#5 Suppose L_1 and L_2 are r.e. Let M_1 and M_2 be TM's such that $L(M_1) = L_1$ and $L(M_2) = L_2$. A third TM M can run both M_1 and M_2 together and enter and enter a final state if either one does (e.g., M can put M_1 & M_2 through their respective moves alternately). Any w which causes either M_1 or M_2 to enter a final state will cause M to enter a final state and vice versa. $\therefore L(M_1) \cup L(M_2) = L(M)$
 $\therefore L_1 \cup L_2$ will be r.e.

#7 Yes. let M_1 and M_2 be as in problem 5 above. Another TM M can be designed to run both together and enter a final state if both M_1 and M_2 do. (For example, M can simulate M_1 and M_2 as follows. Put M_1 & M_2 alternately through their moves and if one them halts in a final state, then continue only with the other one until it halts in a final state.) Then $L(M) = L(M_1) \cap L(M_2)$. So $L_1 \cap L_2$ is also r.e.

#8 If L_1 & L_2 are recursive, then \overline{L}_1 , \overline{L}_2 , L_1 and L_2 will all be r.e. So by problems 5 and 6,

$L_1 \cup L_2$ will be r.e.
and $\overline{L}_1 \cap \overline{L}_2 = \overline{L_1 \cup L_2}$ will be r.e.

Since $L_1 \cup L_2$ & $\overline{L_1 \cup L_2}$ are both r.e., it follows that $L_1 \cup L_2$ is recursive.

Similarly $L_1 \cap L_2$ & $\overline{L_1 \cap L_2} = \overline{L}_1 \cup \overline{L}_2$ will be both r.e. Hence $L_1 \cap L_2$ will be recursive.

Note: recursive sets are closed under \cup , \cap and compliments. r.e. sets are not closed under compliments.

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(27)

$$\begin{aligned}
 \#1. (a) \text{ADD}(3,4) &= \text{ADD}(3,3+1) = \text{ADD}(3,3)+1 \\
 &= \text{ADD}(3,2+1)+1 = (\text{ADD}(3,2)+1)+1 \\
 &= (\text{ADD}(3,1+1)+1)+1 = ((\text{ADD}(3,1)+1)+1)+1 \\
 &= ((\text{ADD}(3,0+1)+1)+1)+1 \\
 &= (((\text{ADD}(3,0)+1)+1)+1)+1 \\
 &= (((3+1)+1)+1)+1 = (((4+1)+1)+1) \\
 &= (5+1)+1 = 6+1 = 7
 \end{aligned}$$

$$\begin{aligned}
 (b) \text{MULT}(2,3) &= \text{MULT}(2,2+1) = \text{MULT}(2,2)+2 \\
 &= \text{MULT}(2,1+1)+2 = (\text{MULT}(2,1)+2)+2 \\
 &= (\text{MULT}(2,0+1)+2)+2 = ((\text{MULT}(2,0)+2)+2)+2 \\
 &= ((0+2)+2)+2 = \dots = 6
 \end{aligned}$$

$$\begin{aligned}
 \#2. \text{Note } 1 \div ((x \div y) + (y \div x)) &= \begin{cases} 1 & \text{if } x=y \\ 0 & \text{if } x \neq y \end{cases} \\
 \therefore \text{equal}(x,y) &= \text{monus}(c_1(p_1(x,y)), \text{add}(\text{monus}(x,y), \text{monus}(p_2(x,y), p_1(x,y)))) \\
 &= \text{monus}(p_1(x,y), \text{mult}(p_1(x,y), \text{equal}(x,y)))
 \end{aligned}$$

#3 (a) Note that $f(x,y) = x \div h(x,y)$ where

$$h(x,y) = \begin{cases} 0 & \text{if } x \neq y \\ x & \text{if } x=y \end{cases}$$

$$\begin{aligned}
 \text{So } f(x,y) &= x \div (x \cdot \text{equal}(x,y)) \\
 &= \text{monus}(p_1(x,y), \text{mult}(p_1(x,y), \text{equal}(x,y)))
 \end{aligned}$$

$$(b) f(0) = 1$$

$$f(y+1) = \text{mult}(y, f(y))$$

From this we can see $f(y) = y!$ is primitive recursive by doing a little work.

#5 (a) $\text{div}(x, y)$ was defined to be the largest integer n such that $x \geq ny$. When $y=0$, $\text{div}(x, y)$ is undefined because there is no largest integer n such that $x \geq n \cdot 0$. So we will amend the definition as follows:

$$\text{div}(x, y) = \begin{cases} 0 & \text{if } y=0 \\ \text{largest } n \text{ such that } x \geq ny \text{ if } y \neq 0. \end{cases}$$

Now $\text{div}(0, y) = 0$

$$\text{div}(x+1, y) = \begin{cases} \text{div}(x, y) & \text{if } x+1 < y(1+\text{div}(x, y)) \\ \text{div}(x, y)+1 & \text{if } x+1 = y(1+\text{div}(x, y)) \end{cases}$$

$\therefore \text{div}(0, y) = 0$

$$\text{div}(x+1, y) = \text{div}(x, y) + \text{equals}(x+1, y + (\text{y} \cdot \text{div}(x, y)))$$

Now this is almost okay except the recursion is being done on the first coordinate. In primitive recursion, we have to do it on the last coordinate.

$$\text{div}(x, y) = f(y, x) = f(p_2(x, y), p_1(x, y))$$

where $f(x, 0) = z(x)$

$$\begin{aligned} f(x, y+1) &= \text{div}(y+1, x) \\ &= \text{div}(y, x) + \text{equals}(y+1, x + x \cdot \text{div}(y, x)) \\ &= f(x, y) + \text{equals}(y+1, x + x \cdot f(x, y)) \\ &= h(x, y, f(x, y)) \end{aligned}$$

where

$$\begin{aligned} h(x, y, z) &= \text{add}(p_3(x, y, z), \text{equals}(s(p_2(x, y, z)), \\ &\quad \text{add}(p_1(x, y, z), \text{mult}(p_1(x, y, z), p_3(x, y, z)))) \end{aligned}$$

$\therefore \text{div}(x, y)$ is primitive recursive.

$$5. (b) \text{rem}(x, y) = x - y \cdot \text{div}(x, y) \\ = \text{monus}(P_1(x, y), \text{mult}(P_2(x, y), \text{div}(x, y)))$$

$$(c) \text{max}(x, y) = x + y - x \\ = \text{add}(P_1(x, y), \text{monus}(P_2(x, y), P_1(x, y)))$$

$$(d) \text{min}(x, y) = x - (x - y) \\ = \text{monus}(P_1(x, y), \text{monus}(P_1(x, y), P_2(x, y)))$$

$$(e) \text{gr}(x, y) = \begin{cases} 1 & \text{if } x > y \\ 0 & \text{if } x \leq y \end{cases} \\ = 1 - \text{equal}(x - y, 0)$$

$$= \text{monus}(C_1(P_1(x, y)), \text{equal}(\text{monus}(x, y), Z(P_1(x, y))))$$

$$(f) f(x) = \lfloor \sqrt{x} \rfloor = (\mu y) ((y+1)^2 > x) \\ = (\mu y) [g(x, y) = 0],$$

where $g(x, y) = 1 - \text{gr}((y+1)^2, x) =$
 $\text{monus}(C_1(P_1(x, y)), \text{gr}(\text{mult}(S(P_2(x, y), S(P_2(x, y))), P_1(x, y))))$

$$(g) f(x) = \lfloor \log_2(x+1) \rfloor = (\mu y) [2^{y+1} > x+1] \\ = (\mu y) [g(x, y) = 0] \text{ where}$$

$$g(x, y) = 1 - \text{gr}(2^{y+1}, x+1) \\ = \text{monus}(C_1(P_1(x, y)), \text{gr}(\text{exp}(C_2 P_1(x, y), S(P_2(x, y))), S(P_2(x, y))))$$

9 (a) $A(1, y) = A(1, (y-1)+1)$
 $= A(1-1, A(1, y-1))$
 $= A(0, A(1, y-1))$
 $= A(1, y-1) + 1$
 $= A(1, (y-2)+1) + 1$
 $= A(0, A(1, y-2)) + 1 + 1$
 $= A(1, y-2) + 1 + 1 + \dots + 1$
 \vdots
 $= A(1, y-y) + \underbrace{1 + 1 + \dots + 1}_{y \text{ times}}$
 $= A(1, 0) + y$
 $= A(0, 1) + y$
 $= (1+1) + y$
 $= y+2.$

6. $f(0) = 2^0 = 1$

$f(n+1) = 2^{n+1} = 2^n + 2^n = \text{Add}(f(n), f(n))$

From this we can see that $f(n)$ is prim. recursive

$f = \text{prim. rec. } (\text{so } 0, \text{ ADD} \circ [I_3^{(3)}, I_3^{(3)}])$

7. $g(x, 0) = x^0 = 1$

$g(x, y+1) = x^{y+1} = x^y \cdot x$

From this we can see that $g(x, y) = x^y$ is prim. recursive

$g = \text{prim. rec. } (\text{so } z, \text{ MULT} \circ [I_3^{(3)}, I_1^{(3)}])$

$$9(a) \quad A(1, y) = y+2 \text{ (done on page 40)}$$

$$\begin{aligned} 9(b) \quad A(2, y) &= A(2, (y-1)+1) = A(2-1, A(2, y-1)) \\ &= A(1, A(2, y-1)) = A(2, y-1) + 2 \text{ by (a)} \\ &= A(2, (y-2)+1) + 2 = A(2-1, A(2, y-2)) + 2 \\ &= A(1, A(2, y-2)) + 2 = (A(2, y-2) + 2) + 2 \text{ by (a)} \\ &= \dots \\ &= A(2, y-y) + \underbrace{2+2+\dots+2}_{y \text{ times}} \\ &= A(2, 0) + 2y = A(1, 1) + 2y \\ &= 1+2+2y \text{ by (a)} \\ &= 2y+3 \end{aligned}$$

$$\begin{aligned} 9(c) \quad A(3, y) &= A(3, (y-1)+1) = A(3-1, A(3, y-1)) \\ &= A(2, A(3, y-1)) = 2A(3, y-1) + 3 \text{ by (b)} \\ &= 2[A(3, y-1) + 3] - 3 \\ &= 2[A(3, (y-2)+1) + 3] - 3 \\ &= 2[A(2, A(3, y-2)) + 3] - 3 \\ &= 2[2A(3, y-2) + 3 + 3] - 3 \text{ by (b)} \\ &= 2 \cdot 2 \cdot [A(3, y-2) + 3] - 3 \\ &= \dots \\ &= \underbrace{2 \cdot 2 \cdot 2 \cdots 2}_{y \text{ times}} \cdot [A(3, y-y) + 3] - 3 \\ &= 2^y [A(3, 0) + 3] - 3 \\ &= 2^y [A(2, 1) + 3] - 3 \\ &= 2^y [(2 \cdot 1 + 3) + 3] - 3 \\ &= (2^y, 8) - 3 = 2^{y+3} - 3. \end{aligned}$$

$$\begin{aligned}
 10. (a) A(4,1) &= A(4,0+1) = A(3, (A(4,0))) \\
 &= 2^{A(4,0)+3} - 3 \quad \text{by 9(c)} \\
 &= 2^{A(3,1)+3} - 3 \\
 &= 2^{2^{1+3}-3+3} - 3 \quad \text{by 9(c)} \\
 &= 2^{18} - 3
 \end{aligned}$$

$$(b) A(4,2) = A(4,1+1) = A(3, A(4,1))$$

$$\begin{aligned}
 &= 2^{A(4,1)+3} - 3 \\
 &= 2^{2^{16}-3+3} - 3 \quad \text{by 10(a)} \\
 &= 2^{2^{16}} - 3
 \end{aligned}$$

11. In general

$$A(4,y) = \underbrace{2^2}_{y \text{ times}} \cdots \underbrace{2^2}_{2^2} \cdots \underbrace{2^2}_{2^2} - 3 = \underbrace{2^2}_{y+3 \text{ times}} \cdots \underbrace{2^2}_{2^2} \cdots \underbrace{2^2}_{2^2} - 3.$$

15 (a) $(\forall y)[g(x,y) = 0]$

= smallest y such that $x \cdot y = 0$

= 0 because $x \cdot 0 = 0$ & 0 is the smallest value of y with $x \cdot y = 0$.
domain = \mathbb{N} .

(b) $(\forall y)[2^x + y - 3 = 0]$

= smallest y such that $2^x + y - 3 = 0$

$$= \begin{cases} 3 - 2^0 & \text{if } x=0 \text{ b.c. } 2^0 + 0 - 3 \neq 0 \text{ & } 2^0 + 1 - 3 \neq 0 \\ 3 - 2^1 & \text{if } x=1 \text{ b.c. } 2^1 + 0 - 3 \neq 0 \\ \text{undefined} & \text{if } x > 1 \end{cases}$$

$$= \begin{cases} 2 & \text{if } x=0 \\ 1 & \text{if } x=1 \\ \text{und} & \text{if } x > 1 \end{cases}$$

domain = {0, 1}. (The textbook is sloppy - it should not use "-". It should use "÷" (minus) but it didn't.)

(c) $(\forall y)\left(\left[\frac{x-1}{y+1}\right] = 0\right)$

$$= \text{und. if } x=0 \text{ because } \left[\frac{-1}{y+1}\right] = -1 \text{ for all } y \neq 0$$

$$\begin{cases} 0 & \text{if } x=1 \text{ because } \left[\frac{0}{0+1}\right] = 0 \\ x-1 & \text{if } x \geq 2 \text{ because } \left[\frac{x-1}{y+1}\right] \neq 0 \text{ for all } y \leq x-2 \end{cases}$$

$$= \begin{cases} \text{und if } x=0 \\ x-1 & \text{if } x > 0 \end{cases} \text{ and } \left[\frac{x-1}{x-1+1}\right] = 0 \text{ for all } x \geq 2$$

domain = $\mathbb{N} - \{0\}$

(d) $(\forall y)(x [\text{mod}(y+1)] = 0) = \begin{cases} 0 & \text{if } x=0 \text{ b.c. } 0 \text{ mod } 1 = 0 \\ 0 & \text{if } x > 0 \text{ b.c. } x \text{ mod } 1 = 0 \end{cases}$

domain = \mathbb{N} . (A rather silly question)

END