

Answer all 6 questions. No calculators, notes, or cell-phones are allowed. An unjustified answer will receive little or no credit. Begin each of the six questions on six separate pages.

- (16) 1. (a) Find an NFA, M , which is *equivalent* to the RLG G given below.

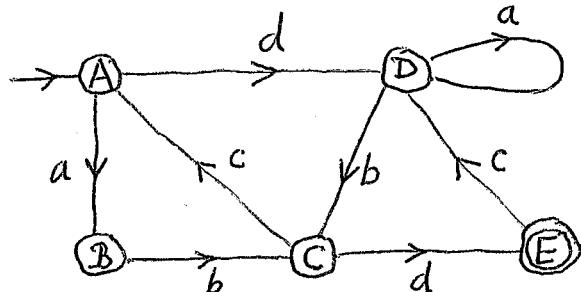
$$G: \quad \rightarrow B, \quad B \rightarrow 10B, \quad B \rightarrow 0C, \quad C \rightarrow 10, \quad C \rightarrow \lambda, \quad C \rightarrow 10D,$$

$$C \rightarrow E, \quad D \rightarrow 0B, \quad E \rightarrow \lambda, \quad E \rightarrow 1C, \quad E \rightarrow 11.$$

- (b) Find an RLG, G , which is equivalent to the NFA in Problem 2(a) below.

- (16) 2. (a) Find a *regular expression* for the language accepted by the NFA M shown on the right.

- (b) Define what is the *busy-beaver function*, $\beta(n)$.



- (16) 3. (a) Define the *initial functions* and the operation called *primitive recursion*.

- (b) Show that $F(x,y) = 3x + 5y + 2$ is a *primitive recursive function* by finding primitive recursive functions g and h such that $F = \text{prec}(g,h)$.

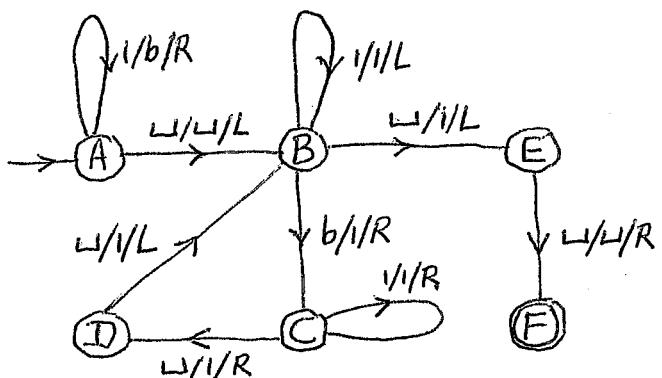
- (16) 4.(a) If $g(x,y)$ is a function from N^2 to N define what is $\mu[g, 0]$ and then define what is a μ -*recursive function* from N^k to N .

- (b) Let $f(x) = \text{Ceiling function of } [x(x+2)]^{1/3}$. Show that f is a μ -*recursive function*. [You may use the fact that PRED, MONUS, ADD, MULT & SIGN are primitive recursive if needed in #4, but you are **not** allowed to do so in Question #3.]

- (18) 5.(a) Define what is a *Turing-computable partial function* f from N to N .

- (b) Show what happens at each step if (i) 1 and (ii) λ are the inputs for the TM, M , shown on the right.

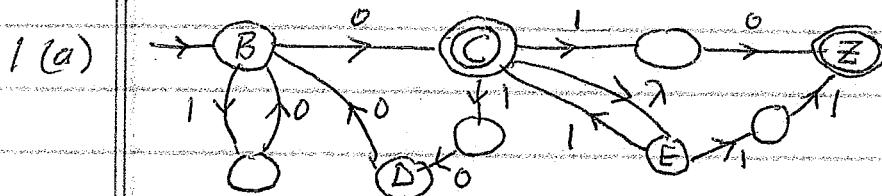
- (c) What is the *function computed* by M in monadic (base 1) notation?



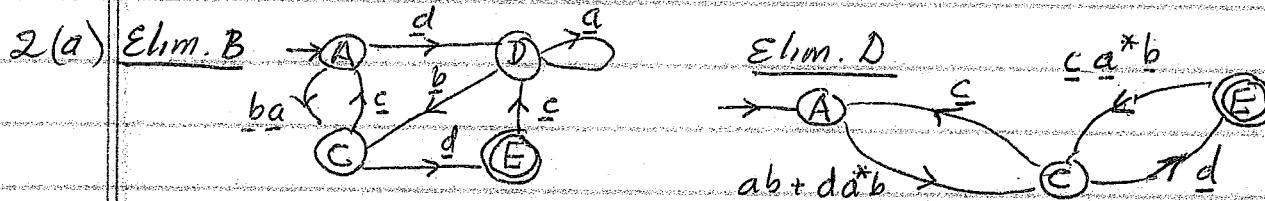
- (18) 6. Determine which of the following languages are regular and which are not.

$$(a) L_1 = \{a^k b^n : k \pmod 3 > (1-n^2) \pmod 3\} \quad (b) L_2 = \{b^k c^n : k > 3+n^2\}.$$

[If you say that it is regular, you must find a regular expression for it; if you say it is non-regular, you must give a complete proof]

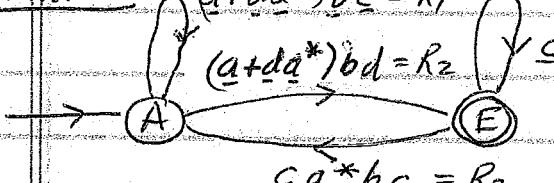


(b) $\rightarrow A, A \rightarrow aB, A \rightarrow dB, B \rightarrow bC, C \rightarrow cA, C \rightarrow dE$
 $E \rightarrow \lambda, E \rightarrow cD, D \rightarrow aD, D \rightarrow bC.$



Elim. C

$$(a+da^*)bc = R_1$$



$$\text{Elim. } D \rightarrow (a+da^*)bc = R_1$$

$$\mathcal{L}(M) = R_1^* R_2 (R_4 + R_3 R_1^* R_2)^*$$

$$\mathcal{L}(M) = ((a+da^*)bc)^* \cdot (a+da^*)bd [ca^*bd + ca^*bc \cdot ((a+da^*)bc)^*] / (a+da^*) \cdot bd]^*$$

3(a) The initial functions are the constant 0, the zero function $z(x)=0$, the successor function $s(x)=x+1$, and the projective functions $I_{k,n}(x_1, \dots, x_n) = x_k$ if $1 \leq k \leq n$, and λ if $k=0$.

Primitive recursion is the operation that takes a function $g: \mathbb{N}^n \rightarrow \mathbb{N}$ and a function $h: \mathbb{N}^{n+2} \rightarrow \mathbb{N}$ and produces a function $f: \mathbb{N}^{n+1} \rightarrow \mathbb{N}$ by putting $f(x, 0) = g(x)$ and $f(x, y, f(x, y)) = h(x, y, f(x, y))$, $x = \langle x_1, \dots, x_n \rangle$

(b) $F(x, y) = 3x + 5y + 2$. So $F(x, 0) = 3x + 2 \leftarrow g(x, y)$

and $F(x, s(y)) = 3x + 5(s(y) + 1) + 2 = (3x + 5y + 2) + 5 \leftarrow h(x, y, f(x, y))$

$\therefore h = s_0 s_0 s_0 s_0 s_0 I_{3,3}$ and $g(x) = 3x + 2$. Now $g(0) = 2$

and $g(s(x)) = 3(x+1) + 2 = (3x + 2) + 3$. $\therefore g = \text{prec}(s_0 s_0 0, s_0 s_0 s_0 I_{2,2})$

$\therefore F = \text{prec}(g, h) = \text{prec}(g, s_0 s_0 s_0 s_0 s_0 I_{3,3})$

$= \text{prec}(\text{prec}(s_0 s_0 0, s_0 s_0 s_0 I_{2,2}), s_0 s_0 s_0 s_0 s_0 I_{3,3})$

4(a) If $f = \mu[g, 0]$ and $g: \mathbb{N}^2 \rightarrow \mathbb{N}$, this means that $f: \mathbb{N} \rightarrow \mathbb{N}$ is the partial function defined by

$$f(x) = \begin{cases} \text{smallest value of } y \text{ such that } g(x, y) = 0 \\ \text{undefined, if } g(x, y) \text{ is never zero for any } y. \end{cases}$$

A μ -recursive function from \mathbb{N}^k to \mathbb{N} is any function that can be obtained from the initial functions by a finite number of applications of compositions, cartesian products, primitive recursions, and minimization on total functions.

(b) Let $g(x, y) = x(x+2) - y^3$. Then $f(x) = (\mu y)[g(x, y) = 0]$

$$\therefore f = \mu[g, 0] = \mu[\text{MONUS} \circ \{\text{MULT} \circ (I_{1,2} \wedge S_0 S_0 \circ I_{1,2}) \wedge \text{MULT} \circ (\text{MULT} \circ (I_{2,2} \wedge I_{2,2}) \wedge I_{2,2}), 0\}]$$

5(a) A partial function $f: \mathbb{N} \rightarrow \mathbb{N}$ is Turing-computable if we can find a TM M such that M halts in an accepting state and outputs $f(n) \Leftrightarrow n \in \text{dom}(f)$.

$$(b) (i) \langle A, 1 \rangle \vdash \langle A, b \sqcup \rangle \vdash \langle B, b \rangle \vdash \langle C, 1 \sqcup \rangle \vdash \langle D, 11 \sqcup \rangle$$

$\vdash \langle B, 111 \rangle \vdash \langle B, 111 \rangle \vdash \langle B, \sqcup 111 \rangle \vdash \langle E, \sqcup 1111 \rangle \vdash \langle F, 1111 \rangle$ halts

$$(ii) \langle A, \sqcup \rangle \vdash \langle B, \sqcup \rangle \vdash \langle E, \sqcup \rangle \vdash \langle F, 1 \rangle \text{ halts}$$

$$(c) \therefore f(0) = 1, f(1) = 4, \text{ & you can check that } f(2) = 7. \text{ So}$$

$f(n) = 3n + 1$ is what we expect. By looking at M , you can see this.

6(a) If $n \equiv 0 \pmod{3}$, then $k \pmod{3} \geq 1 - 0^2 = 1 \pmod{3} \Rightarrow k \equiv 2 \pmod{3}$;

$n \equiv 1 \pmod{3}$, then $k \pmod{3} \geq 1 - 1^2 = 0 \pmod{3} \Rightarrow k \equiv 1 \text{ or } 2 \pmod{3}$.

$n \equiv 2 \pmod{3}$, then $k \pmod{3} \geq 1 - 2^2 = 0 \pmod{3} \Rightarrow k \equiv 1 \text{ or } 2 \pmod{3}$.

So a regular expression for L_1 will be

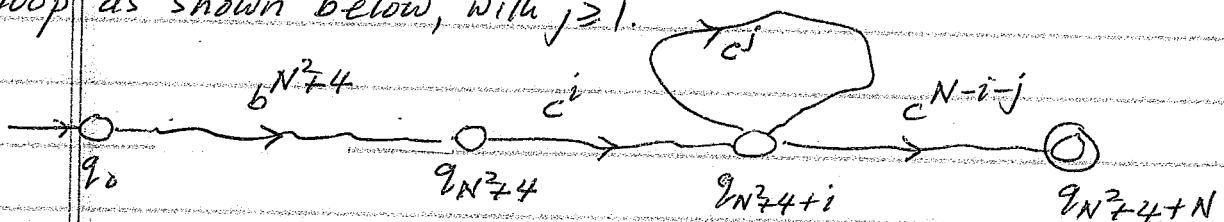
$$aa.(aaa)^*(bbb)^* + (a+aa).(aaa)^*.b.(bbb)^* + (a+aa).(aaa)^*.b.b.(bbb)^*$$

Hence L_1 is a regular language

(b) L_2 is not a regular language. We will give a proof by contradiction. Suppose was a regular language.

6(b) Then we can find a λ -free NFA M_2 such that $L(M_2) = L_2$.

Let $N = \text{number of states in } M_2$ and consider the string $b^{4+N^2}c^N$. since $4+N^2 > 3+(N)^2$, $b^{4+N^2}c^N \in L_2$ and so will be accepted by M_2 . Since it takes $N+1$ states to process the c^N , the acceptance track of $b^{4+N^2}c^N$ must have a loop as shown below, with $j \geq 1$.



Now if we ride this loop twice, we will see that M_2 accepts $b^{N^2+4}c^i c^j c^{N-i-j} = b^{N^2+4}c^{N+j}$. But $N^2+4 \neq 3+(N+j)^2$ because $3+(N+j)^2 = 3+N^2+2Nj+j^3 \geq N^2+6$ since $N, j \geq 1$.

So M_2 accepted a string which is not in L_2 and this contradicts the fact that M_2 was chosen so that $L(M_2) = L_2$. Hence L_2 must be non-regular.

END