MAD 3512 - THEORY OF ALGORITHMS FLORIDA INT'L UNIV.
TEST #2 – SP 2025 TIME: 75 min.

Answer all 6 questions. No calculators, notes, or on-line stuff are allowed. An unjustified answer will receive little or no credit. Draw a line to separate each of your 6 solutions to the 6 questions.

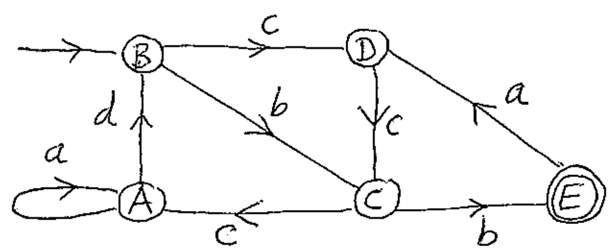
(16) 1. (a) Find an NFA, *M,* which is *equivalent* to the RLG G given below.

$$G: \ \rightarrow A, \quad A \rightarrow 10A, \quad A \rightarrow 0B, \quad A \rightarrow 1E, \quad B \rightarrow \lambda, \quad B \rightarrow 01,$$
$$B \rightarrow 1D, \quad D \rightarrow 0B, \quad D \rightarrow 10, \quad E \rightarrow 0, \quad E \rightarrow 1D.$$

(b) Find an RLG, *G,* which is equivalent to the NFA in Problem **2(a) below.**

(16) 2. (a) Find a *regular expression* for the language accepted by the NFA *M* shown on the right.

(b) Write down completely what the *Halting-problem* says.



(16) 3. (a) Write down the *initial functions* and define what "$F = prec(g,h)$" means.

(b) Show that $F(x,y) = 2x+4y+5$ is a *primitive recursive function* on $\mathbb{N}\times\mathbb{N}$. by finding primitive recursive functions g and h such that $F = prec(g,h)$.

(16) 4.(a) Define what "f is obtained by the *minimization* of the function g: $\mathbb{N}^{n+1} \rightarrow \mathbb{N}$ " means, and define what is a *μ-recursive* partial function on $\mathbb{N}^n$.

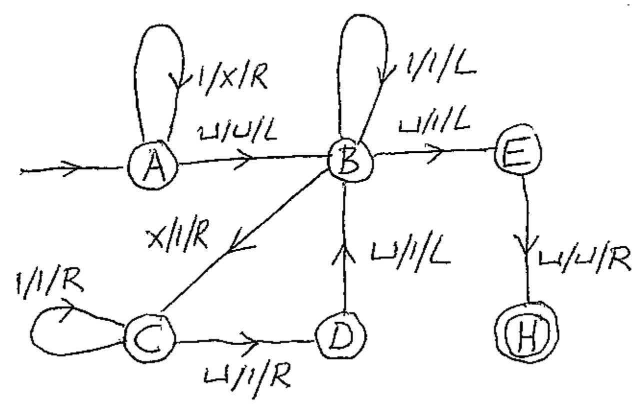
(b) Let $f(x)$ = *Ceiling function* of $[(3x+2)^{1/3}]$. Show that f is a *μ-recursive* function.

[*You may use the fact that PRED, MONUS, ADD, MULT & SIGN are primitive recursive if needed in #4, but you are **NOT** allowed to do so in Question #3.*]

(18) 5.(a) Define what is a *Turing-computable partial function* f: $\mathbb{N}^n \rightarrow \mathbb{N}$.

(b) Show what happens at each step if (i) λ and (ii) *1* are the inputs for the TM *M,* shown on the right.

(c) What is the *function computed* by *M* in *monadic* notation?



(18) 6. Determine which of the following languages are regular and which are not.

(a) $L_1 = \{a^k b^n : k \ (\mathrm{mod}\ 3) > (1 - n^2) \ (\mathrm{mod}\ 3)\}$ (b) $L_2 = \{c^k d^n : k < 2 + n^2\}$.

[*If you say that it is regular, you must find a regular expression for it; if you say it is non-regular, you must give a complete proof.*] END

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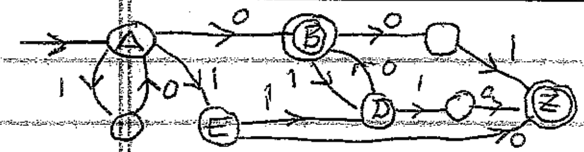
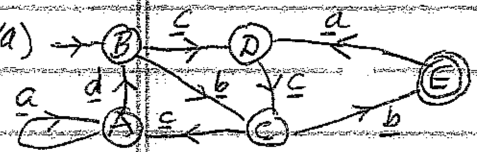
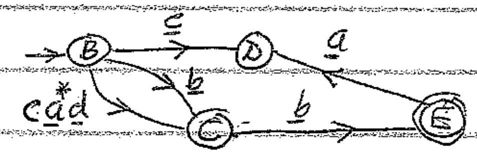
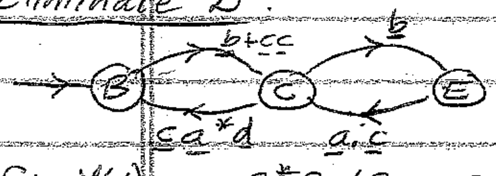
Solutions to Test #2 Spring 2025

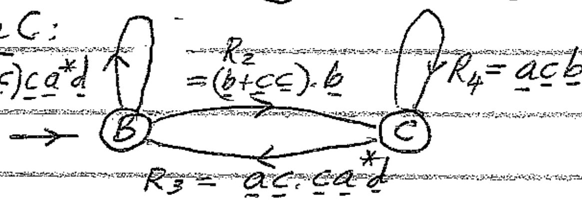
1(a) 

(b) $G: \to B,\ B \to cD,\ B \to bC,\ C \to cA,\ C \to bE$
$A \to aA,\ A \to dB,\ D \to cC,\ E \to aD,\ E \to \lambda$

2(a) 

Eliminate A: 

Eliminate D: 

Eliminate C: 

So $\mathcal{L}(M) = R_1^* R_2 (R_4 + R_3 R_1^* R_2)^*$

$= ((b+cc).ca^*d)^*.((b+cc).b).[acb + (acca^*d).((b+cc).ca^*d)^*.(b+cc)b]^*$

(b) Halting Problem: Is there a TM, H, such for an arb. TM, M, and an arb. input ω for M, H will halt on $c(M)\#c(\omega)$ in an accepting state when M halts on ω; and H will halt in a non-accepting state when M fails to halt on ω. Here $c(M)$ and $c(\omega)$ are codings of M & ω into H's alphabet.

3(a) The initial functions are the zero function, 0, of 0 variables and the zero function, $Z(x) \equiv 0$, of 1 variable, the successor function $S(x) = x+1$, and the projective functions $I_{k,n}(x_1, \dots, x_n) = x_k$ if $1 \le k \le n$ & λ if $k=0$.
$F = prec(g,h)$ is the function, $F: \mathbb{N}^{n+1} \to \mathbb{N}$, produced by putting $F(\underline{x},0) = g(\underline{x})$ & $F(\underline{x}, s(y)) = h(\underline{x}, y, F(\underline{x},y))$. Here $\underline{x} = (x_1, \dots, x_n)$.

(b) $F(x,y) = 2x + 4y + 5$, so $F(x,0) = 2x+5 \leftarrow g(x)$, $g(y) = 2y+5$
& $F(x, s(y)) = 2x + 4(y+1) + 5 = F(x,y) + 4 \leftarrow h(x, y, F(x,y))$
$\therefore\ h = s \circ s \circ s \circ s \circ I_{3,3}$ and since $g(0) = 5$ & $g(s(y)) = g(y) + 2$, we get $\underbrace{g}_{1\ var.} = prec(\underbrace{s \circ s \circ s \circ s \circ s \circ 0}_{0\ var.}, \underbrace{s \circ s \circ I_{2,2}}_{2\ var.})$. Hence $F = prec(\underbrace{g}_{1\ var.}, \underbrace{h}_{3\ var.})$

$= prec(\ prec(s \circ s \circ s \circ s \circ s \circ 0,\ s \circ s \circ I_{2,2}),\ s \circ s \circ s \circ s \circ I_{3,3})$ is a primitive recursive function from $\mathbb{N}^2 \to \mathbb{N}$.

4(a) f is obtained by minimization on $g: \mathbb{N}^{n+1} \to \mathbb{N}$ means that

$$f(\underset{\sim}{x}) = \begin{cases} \text{smallest value of } y \text{ such that } g(x,y)=0; \\ \text{undefined, if } g(x,y)>0 \text{ for every } y \in \mathbb{N}. \end{cases}$$ Here $\underset{\sim}{x} = \langle x_1, \ldots, x_n \rangle$

A μ-recursive partial function on $\mathbb{N}^n$ is any partial function on $\mathbb{N}^n$ that can be obtained from the initial functions by a finite no. of applications of compositions, cartesian products, primitive recursions, and minimizations on total-functions

(b) $f(x) = \lceil (3x+2)^{1/3} \rceil$. Put $g(x,y) = (3x+2) \dot{-} y^3$. Then $f(x) = (\mu y)[g(x,y)=0]$

Now $3x+2 = S(S\{ADD[\underbrace{I_{1,2}(x,y)}_{x}, \underbrace{ADD[I_{1,2}(x,y), I_{1,2}(x,y)]}_{x+x}]\})$

Also $y^3 = MULT\{I_{2,2}(x,y), MULT[I_{2,2}(x,y), I_{2,2}(x,y)]\}$

$\therefore g = MONUS \circ [S \circ S \circ \{ADD \circ (I_{1,2} \wedge ADD \circ (I_{1,2} \wedge I_{1,2}))\}, MULT \circ \{I_{2,2} \wedge MULT \circ (I_{2,2} \wedge I_{2,2})\}]$

$\therefore f = \mu[g, 0]$

$= \mu[MONUS \circ [S \circ S \circ \{ADD \circ (I_{1,2} \wedge ADD \circ (I_{1,2} \wedge I_{1,2}))\}, MULT \circ \{I_{2,2} \wedge MULT \circ (I_{2,2} \wedge I_{2,2})\}, 0]$

5(a) A partial function $f: \mathbb{N}^n \to \mathbb{N}$ is Turing-computable if we can find a TM, M, such that for any input $\underset{\sim}{x} \in dom(f)$, M will halt in an accepting state with output $f(\underset{\sim}{x})$; and for any input $\underset{\sim}{x} \notin dom(f)$, M will halt in a non-accepting state or fail to halt.

(b) (i) $\langle A, \underline{\sqcup} \rangle \vdash \langle B, \underline{\sqcup}\sqcup \rangle \vdash \langle E, \underline{\sqcup} 1 \sqcup \rangle \vdash \langle H, \underline{1} \sqcup \rangle \quad \therefore f(0) = 1$

(ii) $\langle A, \underline{1} \rangle \vdash \langle A, x\underline{\sqcup} \rangle \vdash \langle B, \underline{x} \rangle \vdash \langle C, 1\underline{\sqcup} \rangle \vdash \langle D, 11\underline{\sqcup} \rangle \vdash \langle B, 1\underline{1}1 \rangle$

$\vdash \langle B, \underline{1}11 \rangle \vdash \langle B, \underline{\sqcup}111 \rangle \vdash \langle E, \underline{\sqcup}1111 \rangle \vdash \langle H, \underline{1}111 \rangle \quad \therefore f(1) = 4.$

(c) So $f(0)=1$, $f(1)=4$, & we can check that $f(2)=7$. So by noting that we get 3 extra 1's each time we go through the triangular loop, we can see that the function computed by M is $f(n) = 3n+1$.

6(a) $L_1 = \{a^k b^n : k \pmod 3 > (1-n^2) \bmod 3\}$.

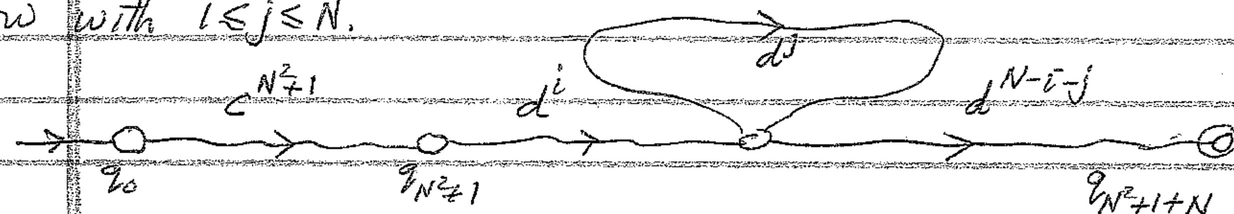
If $n \equiv 0$, then $(1-n^2) \equiv 1-0^2 \equiv 1$, so $k \pmod 3 > 1$ and thus $k \equiv 2 \pmod 3$

If $n \equiv 1$, then $(1-n^2) \equiv (1-1^2) \equiv 0$, so $k \pmod 3 > 0$ & thus $k \equiv 1$ or $2 \pmod 3$

If $n \equiv 2$, then $(1-n^2) \equiv (1-2^2) \equiv 0$, so $k \pmod 3 > 0$ & thus $k \equiv 1$ or $2 \pmod 3$.

$\therefore\ E_1 = aa\cdot(aaa)^*\cdot(bbb)^* + (a+aa)(aaa)^*(bbb)^*b + (a+aa)(aaa)^*\cdot(bbb)^*bb$ is a regular expression that describes L_1. $\therefore L_1$ is a reg. language.

6(b) Suppose L_2 was a regular language. Then we can find a λ-free NFA, M with N states, such that $\mathcal{L}(M) = L_2 = \{c^k d^n : k < 2 + n^2\}$. Now $c^{N^2+1} d^N \in L_2$ because $\underbrace{(N^2+1)}_{k} < 2 + (\underset{n}{N})^2$. So M will accept $c^{N^2+1} d^N$ and its acceptance-track must have a loop as shown below with $1 \le j \le N$.



Now if we skip this loop, then we will see that M will accept the string $c^{N^2+1} d^{N-j}$. But

$$2 + (N-j)^2 = 2 + (N^2 - 2Nj + j^2) = N^2 + 2 + j(j-2N)$$
$$\le N^2 + 2 + 1(1-2) = N^2 + 1 \not< N^2 + 1 \quad \text{bec } j, N \ge 1$$

So $c^{N^2+1} d^{N-j} \notin L_2$ contradicting $\mathcal{L}(M) = L_2$. Hence L_2 cannot be a regular language. END

Extra Stuff

In #4(a) we can also express $3X+2$ by writing:

$$3X+2 = S(S(MULT[S(S(S[Z(I_{1,2}(X,Y)])), I_{1,2}(X,Y)]$$

This will allow us to write

$$\underbrace{g}_{\text{2 variables}} = MONUS \circ \Big[\underbrace{S \circ S \circ MULT \circ \{\underbrace{(S \circ S \circ S \circ Z \circ I_{1,2})}_{3} \wedge \underbrace{I_{1,2}}_{x}\}}_{3X+2},\ \underbrace{MULT \circ \{\underbrace{I_{2,2}}_{y} \wedge MULT \circ (\underbrace{I_{2,2}}_{y} \wedge \underbrace{I_{2,2}}_{y})\}}_{y^3}\Big]$$