

- E1 (a) $A^* = \{a^k : k \geq 0\}$ = set of all strings of a's including the empty string of a's. B (6)
- (b) $A^*B^* = \{a^k b^l : k \geq 0, l \geq 0\}$ = set of all strings of any number of a's followed by any number of b's.
- (c) $(AB)^* = \{(ab)^k : k \geq 0\}$ = set of all strings of any number of (ab)'s.
- (d) $(A \cup B)^* = \{a, b\}^*$ = set of all possible strings of a's and b's in any order.
- (e) $(A \cap B)^* = (\emptyset)^* = \{\lambda\}$ = set consisting of only the empty string.
- (f) $(A \cup B)^* \cdot AB = \{a, b\}^* \cdot \{ab\}$ = set of all strings which are made up of any no. of a's and b's in any order, followed by an (ab).
- (g) $(A^* \cup AB)^* A \cup B^* = \{a, ab\}^* \cdot \{a\} \cup \{b\}^* =$ set of all strings which are made up of any number of b's ; or which are made up of any number of a's or (ab)'s in any order, and followed by an a.
- (h) $(A^* \cup AB)^* - (BAB)^* = \{a, ab\}^* - \{\lambda\}$ = set of all non-empty strings of a's and (ab)'s in any order.

- E2 (a) $\underline{\lambda} \emptyset^* = \{\lambda\} = \underline{\lambda}$ (b) $\underline{\lambda}^* \emptyset^* = \{\lambda\} = \underline{\lambda}$
 (c) $A^* \cup \emptyset^* = A^*$ (d) $(\underline{\lambda} \cup A)^* = A^*$
 (e) $(\emptyset \cup A)^* = A^*$ (f) $(\underline{\lambda}^* \emptyset^*)^* = \{\lambda\}^* = \underline{\lambda}$
 (g) $\{\emptyset, \{\emptyset\}\} - \emptyset = \{\emptyset, \{\emptyset\}\}$ (h) $\{\emptyset, \{\emptyset\}\} - \{\emptyset\} = \{\{\emptyset\}\}$
 (i) $\{\emptyset\} \cap \{\emptyset\} = \{\emptyset\}$ (j) $\{\emptyset\} \cap \emptyset = \emptyset$
 (k) $\underline{\lambda} - \emptyset^* = \emptyset$ (l) $\underline{\lambda} - \{\emptyset^*\} = \{\lambda\} - \{\{\lambda\}\} = \{\lambda\}$
 (m) $\underline{\lambda}^* - \emptyset^* = \emptyset$ (n) \emptyset

- E3 (a) $(0 \cup 1)^* \cdot \underline{101} \cdot (0 \cup 1)^*$ (b) $(\underline{1} \cup \underline{011})^*$ (c) $(\underline{0})^*$
 (d) $(0 \cup \underline{1}) \cdot (\underline{1} \cup \underline{000} \cup \underline{000})^* \cdot (0 \cup \underline{1})$

E.4 (a) Let $\varphi \in A(BA)^*$. Then

B (7)

$$\varphi = \alpha \cdot (\beta_1 \alpha_1) (\beta_2 \alpha_2) \cdots (\beta_k \alpha_k)$$

for some $k \geq 0$ and $\alpha, \alpha_1, \dots, \alpha_k \in A$; $\beta_1, \dots, \beta_k \in B$.

$$\therefore \varphi = \underbrace{(\alpha \cdot \beta_1) (\alpha_1 \cdot \beta_2) \cdots (\alpha_{k-1} \cdot \beta_k)}_{\in (AB)^*} \cdot \underbrace{\alpha_k}_{\in A}$$

Thus $\varphi \in (AB)^* A$, so $A(BA)^* \subseteq (AB)^* A$ (1)

Now let $\varphi \in (AB)^* A$. Then

$$\varphi = (\alpha_1 \beta_1) (\alpha_2 \beta_2) \cdots (\alpha_k \beta_k) \alpha$$

for some $k \geq 0$ and $\alpha, \alpha_1, \dots, \alpha_k \in A$; $\beta_1, \dots, \beta_k \in B$.

$$\therefore \varphi = \underbrace{\alpha_1 (\beta_1 \alpha_2) \cdots (\beta_{k-1} \alpha_k)}_{\in A} \underbrace{(\beta_k \alpha)}_{\in (BA)^*}$$

Thus $\varphi \in A(BA)^*$ and so $(AB)^* A \subseteq A(BA)^*$ (2)

Hence $A(BA)^* = (AB)^* A$ by (1) & (2).

(b) Take $A = \{a\}$ and $B = \{b\}$. Then $a \in A^* B^*$ but $a \notin (AB)^*$. So $(AB)^* \neq A^* B^*$ in general.

(c) Take $A = \{\lambda, a\}$, $B = \{ab\}$, and $C = \{b\}$. Then $ab \in (A \cdot B^*) \cap (A \cdot C^*)$ but $ab \notin A \cdot (B^* \cap C^*)$. So $A \cdot (B^* \cap C^*) \neq (A \cdot B^*) \cap (A \cdot C^*)$ in general.

(d) Let $\varphi \in A^* B$. Then

$$\varphi = \lambda \beta_1 \text{ or } \varphi = \alpha_k \alpha_{k-1} \cdots \alpha_1 \beta$$

for some $k \geq 1$ and $\alpha_1, \dots, \alpha_k \in A$; $\beta, \beta_1 \in B$.

$$\therefore \varphi = \beta_1 \text{ or } \varphi = (\alpha_k \alpha_{k-1} \cdots \alpha_1) \alpha_1 \beta$$

So $\varphi \in B$ or $\varphi \in A^* AB$. $\therefore \varphi \in A^*(AB) \cup B$

Thus $A^* B \subseteq A^*(AB) \cup B$ (1)

Now let $\varphi \in A^*(AB) \cup B$. Then

$$\varphi = \beta_1 \text{ or } \varphi = (\alpha_k \alpha_{k-1} \cdots \alpha_1) \alpha_1 \beta$$

for some $k \geq 0$ and $\alpha, \alpha_1, \dots, \alpha_k \in A$;

E4.(d) So $\varphi = \lambda \cdot \beta$, or $\varphi = (\alpha_k \alpha_{k-1} \dots \alpha_1 \alpha) \cdot \beta$ B ⑧
 In either case $\varphi \in A^*B$. $\therefore A^*(AB) \cup B \subseteq A^*B \dots \dots (2)$
 From (1) & (2) it follows that $A^*B = A^*(AB) \cup B$.

(e) Take $A = \{a\}$ and $B = \{aa\}$. Then $aa \in A^* \cap B^*$
 but $aa \notin (A \cap B)^*$. So $(A \cap B)^* \neq A^* \cap B^*$ in general.

(f) Take $A = \{a\}$, $B = \{b\}$ and $C = \{c\}$. Then
 $ab \in (AB \cup AC)^*$ but $ab \notin A^*(B \cap C)^*$. So
 $A^*(B \cap C)^* \neq (AB \cup AC)^*$ in general.

(g) Let $\varphi \in (A^*B^*)^*$. Then

$$\varphi = \psi_1 \psi_2 \dots \psi_n$$

for some $n \geq 0$ and $\psi_1, \dots, \psi_n \in A^*B^*$. Now each ψ_i is of the form

$$\psi_i = \alpha_1^{(i)} \alpha_2^{(i)} \dots \alpha_{k(i)}^{(i)} \beta_1^{(i)} \beta_2^{(i)} \dots \beta_{l(i)}^{(i)}$$

for some $k(i) \geq 0, l(i) \geq 0$ and $\alpha_1^{(i)}, \dots, \alpha_{k(i)}^{(i)} \in A$
 $\beta_1^{(i)}, \dots, \beta_{l(i)}^{(i)} \in B$.

Thus

$$\varphi = \alpha_1^{(1)} \dots \alpha_{k(1)}^{(1)} \beta_1^{(1)} \dots \beta_{l(1)}^{(1)} \dots \dots \alpha_1^{(n)} \dots \alpha_{k(n)}^{(n)} \beta_1^{(n)} \dots \beta_{l(n)}^{(n)}$$

$\therefore \varphi \in (A \cup B)^*$ and so $(A^*B^*)^* \subseteq (A \cup B)^* \dots \dots (1)$

Now let $\varphi \in (A \cup B)^*$. Then we can split φ into groups of a's followed by b's (0 a's or 0 b's allowed), i.e.

$$\varphi = \psi_1 \psi_2 \dots \psi_n$$

for some $n \geq 0$, where each ψ_i is of the form

$$\psi_i = \alpha_1^{(i)} \alpha_2^{(i)} \dots \alpha_{k(i)}^{(i)} \beta_1^{(i)} \dots \beta_{l(i)}^{(i)} \quad k(i) \geq 0, l(i) \geq 0$$

From this we see that $\psi_i \in A^*B^*$ and so $\varphi \in (A^*B^*)^*$.
 Thus $(A \cup B)^* \subseteq (A^*B^*)^* \dots \dots (2)$. From (1) & (2) it follows that $(A^*B^*)^* = (A \cup B)^*$.

E4(h) Let $\varphi \in A^*(B \cup C)$. Then

$$\varphi = \gamma \cdot x \quad \text{where } \gamma \in A^* \text{ and } x \in B \cup C$$

$$\text{So } \varphi = \gamma \cdot \beta \quad \text{where } \beta \in B, \text{ or}$$

$$\varphi = \gamma \cdot \gamma' \quad \text{where } \gamma' \in C.$$

Thus $\varphi \in A^* \cdot B$ or $\varphi \in A^* \cdot C \quad \therefore \varphi \in (A^* \cdot B) \cup (A^* \cdot C)$

$$\text{Thus } A^*(B \cup C) \subseteq (A^* \cdot B) \cup (A^* \cdot C) \dots \dots \dots (1)$$

Now let $\varphi \in (A^* \cdot B) \cup (A^* \cdot C)$. Then

$$\varphi = A^* \cdot B \quad \text{or} \quad \varphi \in A^* \cdot C.$$

$$\text{So } \varphi = \gamma_1 \cdot \beta \quad \text{where } \gamma_1 \in A^* \text{ and } \beta \in B, \text{ or}$$

$$\varphi = \gamma_2 \cdot \gamma' \quad \text{where } \gamma_2 \in A \text{ and } \gamma' \in C.$$

In either we see that $\varphi \in A^*(B \cup C)$. Thus

$$(A^* \cdot B) \cup (A^* \cdot C) \subseteq A^*(B \cup C) \dots \dots \dots (2)$$

From (1) & (2) it follows that $A^*(B \cup C) = (A^* \cdot B) \cup (A^* \cdot C)$

E5(a) Suppose $S \subseteq (\underline{\omega})^*$ for some $\omega \in V^*$. Then

$$S = \{\omega^{n_i} : i \geq 1\}$$

where $0 \leq n_1 < n_2 < n_3 < \dots < n_i < \dots$ is an increasing sequence of non-negative integers. So if α and β are elements of S , then

$$\alpha = \omega^{n_i} \quad \text{and} \quad \beta = \omega^{n_j}$$

Then

$$\alpha \cdot \beta = \omega^{n_i + n_j} = \omega^{n_j + n_i} = \beta \cdot \alpha$$

Thus S is commutative.

B(10)

(b) Suppose S is commutative. Let w_1 be a string of shortest possible length in S . Then S cannot have another string with length $|w_1|$. For if $w'_1 \in S$ and $|w'_1| = |w_1| = k$, (say), then $w'_1 w_1 = w_1 w'_1$ b.c. S is commutative. \therefore first k letters of w'_1 = first k letters of w_1 . $\therefore w'_1 = w_1$ because $|w'_1| = k$.

Similarly we can show that no two strings in S can have the same length. Let

$$\{w_1, w_2, w_3, \dots\}$$

be strings in S in order of increasing lengths.

Now look at w_1 and w_2 . We know

$$w_1 w_2 = w_2 w_1 \quad \text{and } |w_1| < |w_2|$$

So

w_2 and w_1 must both be powers of some common string. [For example we might have $w_1 = baba$ and $w_2 = bababa$. In this case $w_1 = (ba)^2$ and $w_2 = (ba)^3$.]

Let w be the shortest string such that w_1 and w_2 are powers of w . [For example, we might have $w_1 = aaaa$ and $w_2 = aaaaaa$. In this case $w = a$, (not aa)]

We can then show that each w_k must be a power of w . So we will get

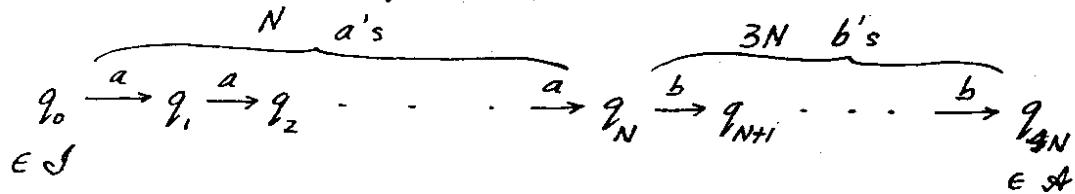
$$w_k = w^{n_k} \quad \text{where } n_k \text{ is some integer.}$$

$$\text{Then } S = \{w^{n_1}, w^{n_2}, w^{n_3}, \dots, w^{n_k}, \dots\}$$

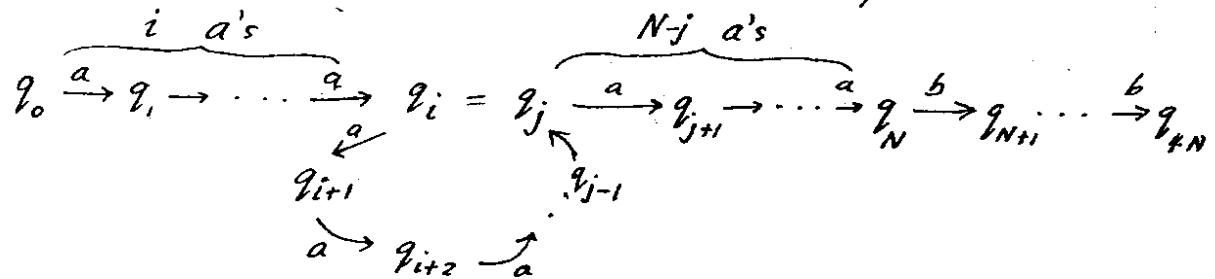
Parts (a) & (b) show that S is commutative if and only if $S \subseteq (\underline{w})^*$ for some $w \in V^*$.

B (11)

E6 (a) Suppose $\{a^k b^{3k} : k \geq 1\} = L$ is regular. Then we can find an FSA M such that $L(M) = L$. Let N be the number of states in M . Then $a^N b^{3N} \in L$ and so it must be accepted by M as follows:



Since there are $N+1$ states in the sequence q_0, \dots, q_N and M has only N states, it follows from the Pigeon Hole Principle that we must have $q_i = q_j$ for some $0 \leq i < j \leq N$. So the situation above is really as shown below:



Let $p = j-i$. Then $p > 0$ and if we traverse the loop r times we see that

$$a^i (a^p)^r a^{N-j} b^{3N} \in L(M).$$

But when $r=0$, this tells us that

$$a^i \cdot \lambda \cdot a^{N-j} b^{3N} = a^{N-(j-i)} b^{3N} = a^{N-p} b^{3N} \in L$$

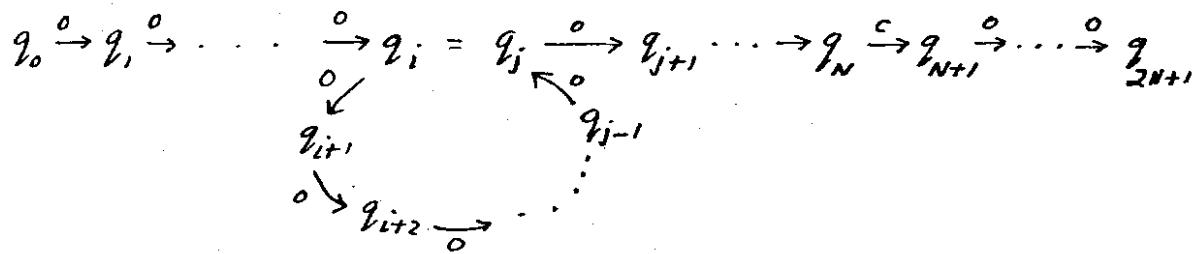
Since $p > 0$, this is a contradiction (because each string in L has 3 times as many b 's as a 's). Hence L is not a regular language.

(b) Suppose $\{w\bar{c}w : w \in \{0,1\}^*\} = L$ is regular. Then we can find an FSA M such that $L(M) = L$. Let N be the number of states in M . Then $0^N \bar{c} 0^N \in L$ and so it must be accepted by M as follows:

$$E6 \text{ (b)} \quad q_0 \xrightarrow{o} q_1 \xrightarrow{o} \dots \xrightarrow{o} q_N \xrightarrow{c} q_{N+1} \xrightarrow{o} q_{N+2} \dots \xrightarrow{o} q_{2N+1} \quad B(12)$$

ϵS ϵA .

Since there are $N+1$ states in the sequence q_0, \dots, q_N we must have (as in 6.1(a)) $q_i = q_j$ for some $0 \leq i < j \leq N$. So the situation above really looks as shown below



Let $p = j - i$.

Then $p > 0$ and if we traverse the loop r times we see that

$o^i(o^p)^r o^{N-j} c o^N \in L(M)$. But when $r=0$, this tells us that

$$o^i \cdot \lambda \cdot o^{N-j} c o^N = o^{N-p} c o^N \in L.$$

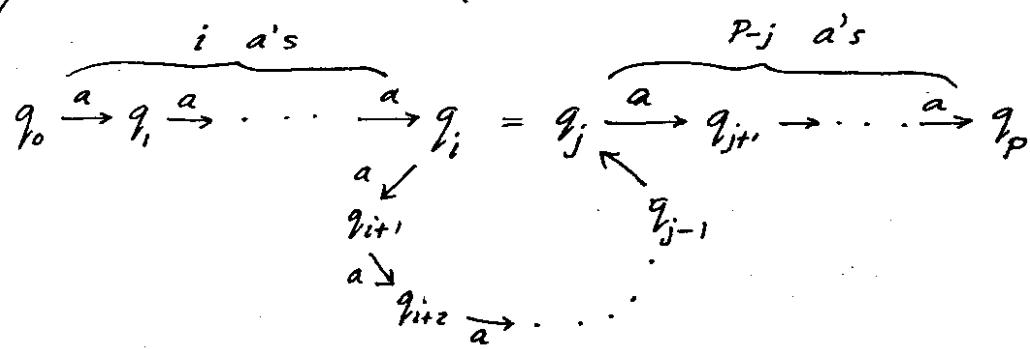
Since $p > 0$, this is a contradiction (because each string in L has the same number of letters before the "c" as after the "c"). Hence L is not a regular language.

E7 Suppose $L = \{a^k : k \text{ is prime}\}$ is regular. (Note: 0 and 1 are not prime). Then we can find an FSA M such that $L(M) = L$. Let N be the number of states in M and let P be a prime number $\geq N$. Then $a^P \in L(M)$ and so it is accepted as shown below:

$$q_0 \xrightarrow{a} q_1 \xrightarrow{a} \dots \xrightarrow{a} q_P$$

Since M has only N states, and there are $P+1 \geq N+1$ states

ET, in the sequence q_0, \dots, q_p it follows, as in 6.1(a)^{B (13)}, that $q_i = q_j$ for some $0 \leq i < j \leq p$. So the situation is really as shown below:



Let $s = j-i$.

Then $s > 0$ and if we traverse the loop r times we see that

$$a^i (a^s)^r a^{p-j} \in L(M) = L.$$

$$\text{So } a^i a^{sr} a^{p-j} = a^{sr} \cdot a^{p-(j-i)} = a^{sr} a^{p-s} \in L.$$

We claim that this leads to a contradiction. There are three cases:

Case (i) : $p-s=0$. In this case $p=s$. If we take $r=2$ we see that $a^{sr} a^{p-s} = a^{2p} \in L$ which is a contradiction because $2p$ is not prime.

Case (ii) : $p-s=1$. In this case, if we take $r=0$, we see that $a^{sr} a^{p-s} = a^1 \in L$ which is a contradiction because 1 is not prime.

Case (iii) : $p-s \geq 2$. In this case, if we take $r=(p-s)$ we see that $a^{sr} a^{p-s} = a^{(s+1)(p-s)} \in L$ which is a contradiction because $(s+1)(p-s)$ is not prime.
[($s+1 \geq 2$ because $s > 0$, and since $(p-s) \geq 2$, we know that $(s+1)(p-s)$ cannot be prime.)]

- E9 (a) non-regular
 (b) regular
 (c) non-regular
 (d) non-regular.

$$\begin{aligned} E8 \text{ (a)} \quad & \{w : w \in X \text{ and } w^R \in Y\} = \{w : w \in X\} \cap \{w : w^R \in Y\} \\ &= \{w : w \in X\} \cap \{w : w \in Y^R\} = X \cap (Y^R). \end{aligned}$$

Since Y is regular, Y^R is regular by the closure theorem.

By the closure theorem again $X \cap (Y^R)$ is regular.

- (b) $\{w : w \in X \text{ and } w^R \notin Y\} = \{w : w \in X\} \cap \{w : w^R \notin Y\}$
 $= \{w : w \in X\} \cap \{w : w^R \in Y^c\} = X \cap \{w : w \in (Y^c)^R\} = X \cap (Y^c)^R.$
 $X \cap (Y^c)^R$ is regular by the Closure theorem as in (a).
- (c) $\{w : w \in X \text{ and } w^R = w\}$ need not be regular. Take $X = \underline{0}^* \underline{1} \underline{0}^*$.
 Then X is regular, and $\{w : w \in X \text{ and } w^R = w\} = \{0^k 1 0^k : k \geq 0\}$
 which can be shown to be non-regular as in 6.1(b).

E10 (a) Yes. $L = \{P0^nQ1^mR : n \geq 1\}$. Take $P = \underline{0}^*$, $Q = \underline{1}$
 and $R = \underline{1}^*$. Then $L = \underline{0}^* \underline{0} \underline{1} \underline{1} \underline{1}^*$ which is regular.

- (b) $\{0^n 1^n : n \geq 1\} \subseteq \underline{0}^* \underline{0} \underline{1} \underline{1}^* \subseteq \{1^k 0^m 1^n 0^k : k \geq 0, m, n \geq 1\}$
 (c) $\underline{0} \underline{1}^* \underline{0} \subseteq \{0^k 1^n 0^k : k \geq 1, n \geq 0\} \subseteq \underline{0}^* \underline{0} \underline{1}^* \underline{0} \underline{0}^*$.

E11 (a) $A - B$ may be regular. Take $A = B = \{a^n b^n : n \geq 1\}$. Then $A - B = \emptyset$

(b) $A \cup B$ may be regular. Take $A = \{a^n b^n : n \geq 0\}$ and
 $B = \{a^i b^j : i \neq j; i, j \geq 0\}$. Then $A \cup B = \underline{a}^* \underline{b}^*$.

(c) $A \cdot B$ may be regular. Take $A = \{a^p : p \text{ is prime}\}$ and
 $B = \{a^q : q \text{ is not prime}\}$. The A and B are non-regular.

But $A \cdot B = \{a^{p+q} : p \text{ is prime and } q \text{ is not prime}\} = \underline{a} \underline{a} \underline{a}^*$
 because any $n \geq 2$ is a sum of a prime and a non-prime.

$$2 = 2 + 0, \quad 3 = 3 + 0, \quad 4 = 3 + 1, \quad 5 = 5 + 0;$$

If $n \geq 6$ and is even, $n = 2 + n-2$ (2 is prime, $n-2$ is non-prime);

If $n \geq 6$ and is odd, $n = 3 + n-3$ (3 is prime, $n-3$ is non-prime).

$$E12. (a) \text{ sign} = \text{PRIM.REC.} [0, s \circ z \circ I_1^{(2)}] \quad s(x) = x+1$$

$$(b) \overline{\text{sign}} = \text{PRIM.REC.} [s(0), z \circ I_1^{(2)}] \quad z(x) = 0 \quad \text{for all } x$$

$$E13 (i) \text{ exp} = \text{PRIM.REC.} [s \circ z, \text{MULT} \circ [I_1^{(3)}, I_3^{(3)}]]$$

$$\text{MULT} = \text{PRIM.REC.} [z, \text{ADD} \circ [I_1^{(3)}, I_3^{(3)}]]$$

$$\text{ADD} = \text{PRIM.REC.} [I_1^{(1)}, s \circ I_3^{(3)}]$$

$$(ii) \text{ abs} = \text{ADD} \circ [\text{MONUS}, \text{MONUS} \circ [I_2^{(2)}, I_1^{(2)}]]$$

$$\text{MONUS} = \text{PRIM.REC.} [I_1^{(1)}, \text{PRED} \circ I_3^{(3)}]$$

$$\text{PRED} = \text{PRIM.REC.} [0, I_1^{(2)}]$$

$$(iii) \text{ Zer} = \overline{\text{sign}}$$

$$(iv) \text{ MIN} = \text{MONUS} \circ [I_1^{(2)}, \text{MONUS}]$$

$$(v) \text{ MAX} = \text{ADD} \circ [I_2^{(2)}, \text{MONUS}]$$

$$(vi) \text{ REM} = \text{PRIM.REC.} [z, \text{MULT} \circ [s \circ I_3^{(3)}, \text{sign} \circ \text{MONUS} \circ [I_2^{(3)}, s \circ I_3^{(3)}]]]$$

$$(vii) \text{ QVO} = \text{PRIM.REC.} [z, \text{ADD} \circ [I_3^{(3)}, \overline{\text{sign}} \circ \text{MONUS} \circ [I_2^{(3)}, s \circ \text{REM}]]]$$

$$(viii) \text{ EQU} = \overline{\text{sign}} \circ \text{abs}$$

$$E14. (a) \text{ ls} = \text{sign} \circ \text{MONUS} \circ [I_2^{(2)}, I_1^{(2)}]$$

$$(b) \text{ gr} = \text{sign} \circ \text{MONUS}$$

$$I_k^{(n)}(x_1, \dots, x_n) = x_k \quad n \geq 1, \quad 1 \leq k \leq n$$

A constant is a function of 0 variables. 0 is an initial function.