MAC 2312 - Calculus II — Florida International Univ.

Test #1 - Spring 2012 — TIME: 100 min.

Answer all 8 questions. An unjustified answer will receive little or no credit. No calculators, formula sheets or cell-phones are allowed. BEGIN EACH QUESTION ON A SEPARATE PAGE.

(12) 1. (a) Evaluate $\int_0^3 (x^2+2x)\,dx$ by finding the Riemann sum $\sum_{k=1}^{n} f(x_k^*)\Delta x_k$ and letting $n \to \infty$. Here $\langle x_0, x_1, \dots, x_n \rangle$ is the uniform partition of $[0,3]$ and x_k^* = the right end-point of each sub-interval.

(b) Check your answer by using anti-derivatives

(12) 2. (a) Write down what the two parts of the Fundamental Theorem of Calculus say.

(b) Find, with justification, $\frac{d}{dx}\left[\int_x^{\tan x} \sqrt{t^2+1}\,dt\right]$.

(12) 3. A particle moves along the x-axis with velocity given by $v(t) = (16t - 4t^3)$ m/s.

a) Find the average velocity from $t = 0$ sec. to $t = 3$ sec.

b) Find the average speed between $t = 0$ sec. & $t = 3$ sec.

(14) 4. Let R be the region bounded by the curves $y = 10\sqrt{x} + 3$, $y = 0$, $x = 1$, and $x = 0$. First make a sketch and then find the volume of the solid formed by

(a) revolving R about the x-axis,

(b) revolving R about the y-axis.

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(12) 5(a) Evaluate $\int_0^3 (x/4)\cdot\sqrt{x+1}\,.dx$ by using the change of variable theorem.

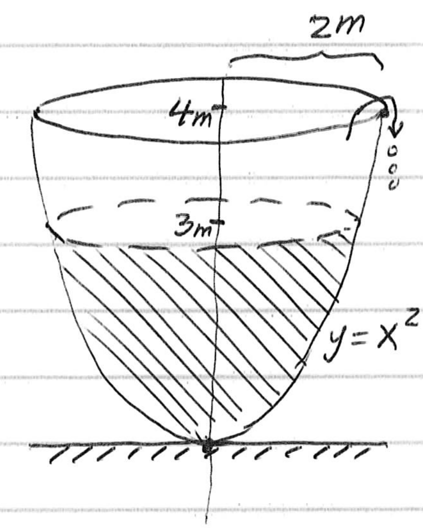
(b) Prove that $\ln(1/a) = -\ln(a)$ from the definition of $\ln(x)$.

(14) 6(a) Find the area enclosed by the curves $y = 2\sqrt{x}$ and $y = x\sqrt{x}$.

(b) Find the length of the curve $y = \frac{x^2}{2} - \frac{\ln(x)}{4}$ as x varies from e^{-1} to e.

(12] 7. Let C be the portion of the curve $y = x^2$ as x varies from 0 to $\sqrt{2}$. Make a sketch of the curved surface S that is formed by revolving C about the y-axis. Then find the surface area of S.

(12) 8. A parabolic cistern of height 4m was formed by revolving the curve $y = x^2$ about the y-axis. If the cistern is filled with water to a depth of 3m, how much work must be done to pump all this water over the rim of the cistern?



[Use $g = 10\,ms^{-2}$ & density of water $= 10^3\ kg/m^3$.].

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END.

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Solutions to Test #1 | Spring 2012

1(a) $\Delta x = (b-a)/n = (3-0)/n = 3/n$, $[a,b] = [0,3]$, $f(x) = x^2+2x$.

$x_k = a + k\Delta x = 0 + k\cdot(3/n) = 3k/n$, $x_k^* = x_k = 3k/n$.

$$\sum_{k=1}^{n} f(x_k^*)\cdot\Delta x_k = \sum_{k=1}^{n}\left\{(3k/n)^2 + 2(3k/n)\right\}\cdot\frac{3}{n} = \frac{3}{n}\sum_{k=1}^{n}\left(\frac{3}{n}\right)^2\cdot k^2 + 2\left(\frac{3}{n}\right)\cdot k$$

$$= \frac{3}{n}\cdot\left(\frac{3}{n}\right)^2\cdot\sum_{k=1}^{n} k^2 + \frac{3}{n}\cdot 2\cdot\frac{3}{n}\cdot\sum_{k=1}^{n} k = \frac{9}{n^2}\cdot\frac{3}{n}\cdot\frac{n(n+1)(2n+1)}{6} + \frac{9}{n^2}\cdot 2\cdot\frac{n(n+1)}{2}$$

$$= \frac{9}{n^2}\,\frac{(n+1)(2n+1)}{2} + \frac{9(n+1)}{n} = \frac{9}{2}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) + 9\left(1+\frac{1}{n}\right)$$

$$\rightarrow \frac{9}{2}(1+0)(2+0) + 9(1+0) = 18 \quad \text{as } n\to\infty$$

(b) $\int_0^3 (x^2+2x)\,dx = \left[\int (x^2+2x)\,dx\right]_0^3 = \left[\frac{x^3}{3} + x^2\right]_0^3 = (9+9) - 0 = 18$ ✓

2(a)(i) If f is continuous on $[a,b]$ and F is any anti-derivative of f over $[a,b]$, then $\int_a^b f(x)\,dx = F(b) - F(a)$.

(ii) If f is continuous on (b,c) and a is any fixed point in (b,c), then $\frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x)$ for each x in (b,c).

(b) $$\frac{d}{dx}\left[\int_x^{\tan x}\sqrt{t^2+1}\,dt\right] = \frac{d}{dx}\left[\int_x^0 \sqrt{t^2+1}\,dt + \int_0^{\tan x}\sqrt{t^2+1}\,dt\right]$$

Put $u = \tan x$

Then $\frac{du}{dx} = \sec^2 x$

$\tan^2 x + 1 = \sec^2 x$

$$= \frac{d}{dx}\left[-\int_0^x \sqrt{t^2+1}\,dt + \int_0^u \sqrt{t^2+1}\,dt\right]$$

$$= -\frac{d}{dx}\int_0^x \sqrt{t^2+1}\,dt + \frac{d}{dx}\left[\int_0^u \sqrt{t^2+1}\,dt\right]$$

$$= -\sqrt{x^2+1} + \frac{d}{du}\left[\int_0^u \sqrt{t^2+1}\,dt\right]\cdot\frac{du}{dx}$$

$$= -\sqrt{x^2+1} + \sqrt{u^2+1}\cdot\sec^2 x = \sec^3 x - \sqrt{x^2+1}.$$

3(a) Average velocity = displacement / (time taken)

$$= \frac{1}{3-0}\int_0^3 (16t - 4t^3)\,dt = \frac{1}{3}\cdot\left[8t^2 - t^4\right]_0^3$$

$$= \frac{1}{3}\cdot[8(9) - 81] = -\frac{9}{3} = -3 \text{ m/s}.$$

3(b) Average speed = (total distance travelled)/time taken.

Now $v(t) = 16t - 4t^3 = 4t(4-t^2) = 4t(2-t)(2+t)$

$$\therefore \quad |v(t)| = \begin{cases} 16t - 4t^3 & 0 \le t \le 2 \\ 4t^3 - 16t & 2 < t \le 3. \end{cases}$$

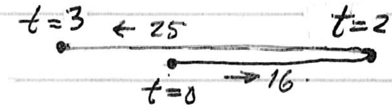
Average speed $= \frac{1}{3-0}\int_0^3 |v(t)|\,dt = \frac{1}{3}\left\{\int_0^2 |v(t)|\,dt + \int_2^3 |v(t)|\,dt\right\}$

$= \frac{1}{3}\left\{\int_0^2 (16t - 4t^3)\,dt + \int_2^3 (4t^3 - 16t)\,dt\right.$

$= \frac{1}{3}\left\{\left[8t^2 - t^4\right]_0^2 + \left(t^4 - 8t^2\right]_2^3\right\}$

$= \frac{1}{3}\left\{(32-16) + (81-72) - (-32)\right\}$

$= \frac{1}{3}\left\{16 + 9 + 16\right\} = \frac{1}{3}(41) = \frac{41}{3}$ m/s.

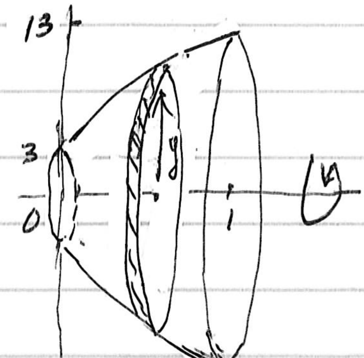


4(a) $V_1 = \int_0^1 \pi . y^2\,dx = \pi\int_0^1 (10\sqrt{x}+3)^2\,dx$

$= \pi\int_0^1 (100x + 60\sqrt{x} + 9)\,dx$

$= \pi\left[\frac{100x^2}{2} + 60.\frac{2}{3}.x^{3/2} + 9x\right]_0^1$

$= \pi(50 + 40 + 9) = 99\pi.$



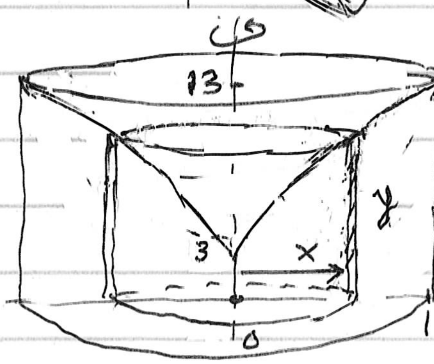
(b) $V_2 = \int_0^1 2\pi x.y.dx$

$= \pi\int_0^1 2.x.(10\sqrt{x}+3)\,dx$

$= \pi\int_0^1 (20x^{3/2} + 6x)\,dx$

$= \pi.\left[20.\frac{2}{5}x^{5/2} + \frac{6x^2}{2}\right]_0^1$

$= \pi(8+3) = 11\pi$



5(a) $\int_0^3 (x/4)\sqrt{x+1}.dx = \int_1^4 (u-1)/4.\sqrt{u}.du$

Put $u = x+1$, $du = dx$
$x=0 \Rightarrow u = 0+1 = 1$
$x=3 \Rightarrow u = 3+1 = 4$

$= \frac{1}{4}\int_1^4 (u-1).u^{1/2}.du$

$= \frac{1}{4}\int_1^4 u^{3/2} - u^{1/2}\,du$

$= \frac{1}{4}\left[\frac{2}{5}u^{5/2} - \frac{2u^{3/2}}{3}\right]_1^4$

$= \left[\frac{2u^{3/2}}{4}\left\{\frac{u}{5} - \frac{1}{3}\right\}\right]_1^4 = \left[\frac{u^{3/2}}{2}\left\{\frac{3u-5}{15}\right\}\right]_1^4$

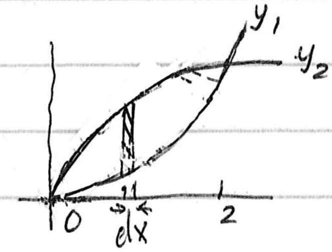
$= \frac{1}{30}\left[u^{3/2}(3u-5)\right]_1^4 = \frac{1}{30}\left[8(7) - 1(-2)\right] = \frac{58}{30} = \frac{29}{15}.$

5(b) $\ln\left(\frac{1}{a}\right) = \int_1^{1/a} \frac{1}{t}\,dt = \int_1^a u \cdot \frac{-1}{u^2}\,du$

Put $u = 1/t$. Then
$t = 1/u$, so $dt = (-1/u^2)\,du$
$t = 1 \Rightarrow u = 1/1 = 1$
$t = 1/a \Rightarrow u = 1/(1/a) = a$

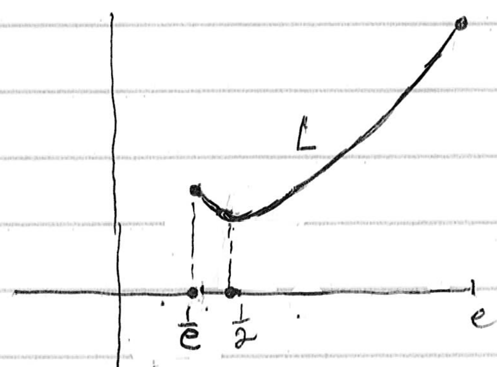
$$= \int_1^a -\frac{1}{u}\,du$$
$$= -\int_1^a \frac{1}{u}\,du$$
$$= -\int_1^a \frac{1}{t}\,dt = -\ln(a).$$

6(a) Let $y_2 = 2\sqrt{x}$ & $y_1 = x\sqrt{x}$. $y_1 = y_2 \Rightarrow 2\sqrt{x} = x\sqrt{x} \Rightarrow$
$\sqrt{x}(2-x) = 0 \Rightarrow x = 0$ or $x = 2$.



$$\text{Area} = \int_0^2 (y_2 - y_1)\,dx$$
$$= \int_0^2 \left(2x^{1/2} - x^{3/2}\right)dx$$
$$= \left[2\cdot\frac{2}{3}x^{3/2} - \frac{2}{5}x^{5/2}\right]_0^2$$
$$= \left[2x^{3/2}\left(\frac{2}{3} - \frac{x}{5}\right)\right]_0^2 = \left[2\cdot 2\sqrt{2}\left(\frac{2}{3} - \frac{2}{5}\right)\right] - [0]$$
$$= 4\sqrt{2}\left(\frac{10-6}{15}\right) = 4\sqrt{2}\cdot\frac{4}{15} = \frac{16\sqrt{2}}{15}.$$

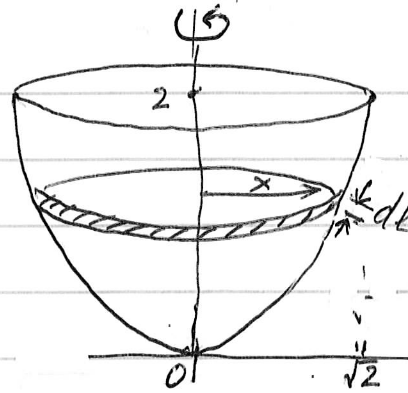
(b) $L = \int_{e^{-1}}^{e} \sqrt{1 + (dy/dx)^2}\,dx$



$$y = \frac{x^2}{2} - \frac{\ln x}{4}$$
$$\frac{dy}{dx} = \frac{2x}{2} - \frac{1/x}{4} = x - 1/(4x)$$
$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \left(x - \frac{1}{4x}\right)^2$$
$$= 1 + x^2 - 2\cdot x\cdot\frac{1}{4x} + \left(\frac{1}{4x}\right)^2$$
$$= 1 + x^2 - 1/2 + \left(\frac{1}{4x}\right)^2$$
$$= x^2 + 1/2 + (1/4x)^2 = \{x + 1/(4x)\}^2$$
$$\therefore L = \int_{e^{-1}}^{e} \sqrt{\left(x + \frac{1}{4x}\right)^2} = \int_{e^{-1}}^{e} \left(x + \frac{1}{4x}\right)dx$$
$$= \left[\frac{x^2}{2} + \frac{\ln(x)}{4}\right]_{e^{-1}}^{e} = \frac{1}{4}\left[2x^2 + \ln x\right]_{e^{-1}}^{e}$$
$$= \frac{1}{4}\left\{(2e^2 + 1) - (2e^{-2} - 1)\right\} = (2e^2 + 2 + 2e^{-2})/4$$
$$= (e^2 + 1 + e^{-2})/2.$$

7. $S = \int_0^{\sqrt{2}} 2\pi . x . \sqrt{1 + (dy/dx)^2} . dx$

$= \int_0^{\sqrt{2}} 2\pi x . \sqrt{1 + (2x)^2}\ dx$



$\boxed{y = x^2,\ \frac{dy}{dx} = 2x}$

$\boxed{\begin{array}{l} \text{Put } u = 1 + 4x^2 \\ du = 8x\,dx \\ du/8 = x\,dx \\ x = 0 \Rightarrow u = 1 \\ x = \sqrt{2} \Rightarrow u = 9 \end{array}}$

$\boxed{dl = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx}$

$= \int_1^9 2\pi . \sqrt{u} . \frac{du}{8}$

$= \frac{\pi}{4} \int_1^9 u^{1/2}\ du$

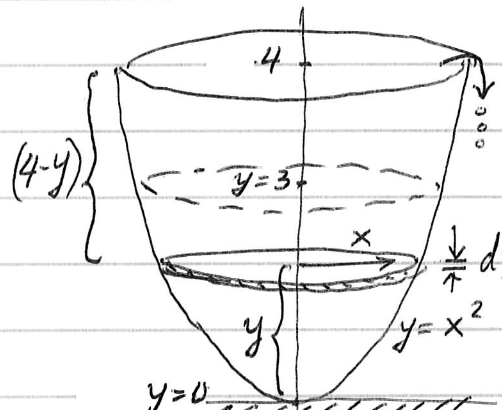
$= \frac{\pi}{4} . \frac{2}{3} \left[u^{3/2}\right]_1^9$

$= \frac{\pi}{6} . (9\sqrt{9} - 1) = 26\pi/6 = 13\pi/3$.

8. $dW = (dm) . g . \text{distance}$

$= (dV)(\text{density}) . g . (4 - y)$

$= \pi . r^2 . dy . 10^3 . 10 . (4 - y)$



So $W = \int_{y=0}^{3} dW$

$= \int_0^3 \pi . 10^4 . (4 - y) . r^2 . dy$

$= \int_0^3 \pi . 10^4 . (4 - y) . x^2 . dy$

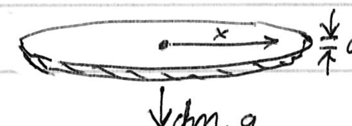
$= \pi . 10^4 \int_0^3 (4 - y) . y . dy$

$= \pi . 10^4 \int_0^3 (4y - y^2)\ dy$

$= \pi . 10^4 . \left[2y^2 - \frac{y^3}{3}\right]_0^3$

$= \pi . 10^4 . (18 - 27/3) = 9\pi . 10^4\ J = 90\pi\ kJ$.

radius, $r = x$



$dV = \pi . x^2 . dy$