MAC 2312 - Calculus II

Florida International Univ.

TEST # 2 - Spring 2012

TIME: 100 min.

Answer all 8 questions. An unjustified answer will receive little or no credit. Always simplify your answers as far as possible. No calculators, formula sheets, or cell-phones are allowed. BEGIN EACH QUESTION ON A SEPARATE PAGE.

(12) 1. Find (a) $\int \frac{1}{x^2} \cdot \ln(x)\, dx$ (b) $\int \frac{x+5}{x^2-2x-3}\, dx$

(14) 2. Find (a) $\int \frac{dx}{\sqrt{x^2+4}}$ (b) $\int \sqrt{4-x^2}\, dx$

(12) 3. Find the value to which the following infinite series converge

(a) $\sum_{k=2}^{\infty} (-1)^k \cdot \frac{3^{k+2}}{5^{k+1}}$ (b) $\sum_{k=2}^{\infty} \frac{1}{k(k+1)}$

(12) 4. The function $f(x)$ has values as shown below. Find

(a) the Trapezoidal Approximation T_6 of $\int_{0.2}^{2.0} f(x)\,dx$,

(b) the Simpson Parabolic Approx. S_6 of $\int_{0.2}^{2.0} f(x)\,dx$.

x	0.2	0.5	0.8	1.1	1.4	1.7	2.0
$f(x)$	3	3	5	4	2	3	1

PLEASE TURN OVER

(12) 5(a) Write down the error estimates for the Trapezoidal and Simpson Approximations of $\int_0^6 f(x)dx$ when $[a,b]$ is split into n equal subintervals. [Say what are K_2 and K_4].

(b) Evaluate the integral $\int \frac{x+2}{x^2-2x+5}\,dx$

(12) 6. Let $\{a_n\}_{n=1}^{\infty}$ be the sequence defined as follows:

$a_1 = 4$ and $a_n = 1 + (6/a_{n-1})$ for $n \geq 2$.

(a) Find a_2, a_3, and a_4

(b) Assuming a_n converges to L, find the value of L.

(14) 7. First make a sketch of the region involved and then find the value of the following improper integrals.

(a) $\int_1^2 \frac{1}{\sqrt{x-1}}\,dx$ (b) $\int_e^{+\infty} \frac{dx}{x.(\ln x)^2}$

(12) 8(a) Show that for any integer n with $n \geq 2$

$$\int \cos^n x \,.\, dx = \frac{\sin x \,.\, \cos^{n-1} x}{n} + \frac{n-1}{n}\int \cos^{n-2} x\, dx.$$

(b) Hence, or otherwise, find $\int \cos^4 x . dx$.

END

MAC 2312 _ SP 12 _ T2.

1(a) $\int \frac{1}{x^2}\cdot \ln(x)\,dx = \int \ln x \cdot x^{-2}dx = \int u \cdot dv = uv - \int v\,du$

Put $u = \ln(x)$ & $dv = x^{-2}dx$
Then $du = x^{-1}dx$ & $v = -x^{-1}$

$= -x^{-1}\ln(x) - \int -x^{-1} \cdot x^{-1}dx$

$= -x^{-1}\ln(x) + \int x^{-2}dx$

$= -x^{-1}\ln(x) - x^{-1} + C$

(b) Suppose $\frac{x+5}{(x-3)(x+1)} = \frac{A}{x-3} + \frac{B}{x+1}$. Then

$$x+5 = A(x+1) + B(x-3)$$

Putting $x=3$ gives us $3+5 = A(3+1) \Rightarrow A = 2$.
Putting $x=-1$ gives us $-1+5 = B(-1-3) \Rightarrow B = -1$.

$\therefore \int \frac{x+5}{x^2-2x-3}dx = \int \frac{x+5}{(x-3)(x+1)}dx = \int\left(\frac{2}{x-3} - \frac{1}{x+1}\right)dx$

$= 2\ln(x-3) - \ln(x+1) + C = \ln\{(x-3)^2/(x+1)\} + C$.

2(a) $\int \frac{dx}{\sqrt{x^2+4}} = \int \frac{2\sec^2\theta\, d\theta}{2\sec\theta} = \int \sec\theta\, d\theta$

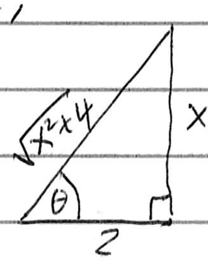
Put $x = 2\tan\theta$. Then
$dx = 2\sec^2\theta\, d\theta$ and
$\sqrt{x^2+4} = \sqrt{4\tan^2\theta + 4}$
$= \sqrt{4\sec^2\theta} = 2\sec\theta$

$= \ln(\sec\theta + \tan\theta) + C$

$= \ln\left(\frac{\sqrt{x^2+4}}{2} + \frac{x}{2}\right) + C$

$= \ln\left(\sqrt{x^2+4} + x\right) + C_1$

where $C_1 = C - \ln(2)$.



(b) $\int \sqrt{4-x^2}\cdot dx = \int 2\cos\theta \cdot 2\cos\theta\, d\theta$

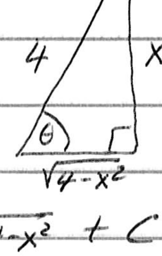
Put $x = 2\sin\theta$. Then
$dx = 2\cos\theta\, d\theta$ and
$\sqrt{4-x^2} = \sqrt{4-4\sin^2\theta}$
$= \sqrt{4\cos^2\theta} = 2\cos\theta$

$= \int 4\cos^2\theta\, d\theta$

$= \int 4\cdot\frac{1}{2}(1+\cos 2\theta)\, d\theta$

$= 2\theta + \sin(2\theta) + C$

$= 2\theta + 2\sin\theta\cos\theta + C$



$= 2\sin^{-1}\left(\frac{x}{2}\right) + 2\cdot\frac{x}{2}\cdot\frac{\sqrt{4-x^2}}{2} + C = 2\sin^{-1}\left(\frac{x}{2}\right) + \frac{x}{2}\sqrt{4-x^2} + C$.

3(a) $\displaystyle\sum_{k=2}^{\infty} (-1)^k \cdot \frac{3^{k+2}}{5^{k+1}} = \sum_{j=0}^{\infty} (-1)^{j+2} \cdot (3)^{j+4}/5^{j+3}$

$\boxed{\begin{array}{l}\text{Put } j = k-2 \\ \text{Then } k = j+2\end{array}}$ $\displaystyle = \sum_{j=0}^{\infty} (-1)^2 \cdot \frac{3^4}{5^3} \cdot \frac{(-1)^j \cdot 3^j}{5^j}$

$\displaystyle = \sum_{j=0}^{\infty} \frac{81}{5(25)} \cdot \left(\frac{-3}{5}\right)^j = \frac{81}{5(25)} \cdot \frac{1}{1-(-3/5)} = \frac{81}{\cancel{5}(25)} \cdot \frac{\cancel{5}}{8} = \frac{81}{200}.$

(b) First $\displaystyle \frac{1}{k(k+1)} = \frac{A}{k} + \frac{B}{k+1} \Rightarrow 1 = A(k+1) + Bk$

Putting $k=0$ gives $1 = A$. Putting $k=-1$ gives $B=-1$.

$\displaystyle \therefore\ S_n = \sum_{k=2}^{n} \frac{1}{k(k+1)} = \sum_{k=2}^{n} \frac{1}{k} - \frac{1}{k+1}$

$\displaystyle = \frac{1}{2} + \cancel{\frac{1}{3}} + \cancel{\frac{1}{4}} + \cancel{\frac{1}{5}} + \cdots + \cancel{\frac{1}{n}}$
$\displaystyle \quad - \cancel{\frac{1}{3}} - \cancel{\frac{1}{4}} - \cancel{\frac{1}{5}} - \cdots - \cancel{\frac{1}{n}} - \frac{1}{n+1}$

$\displaystyle = \frac{1}{2} - \frac{1}{n+1}. \quad \therefore \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{1}{2} - \frac{1}{n+1} = \frac{1}{2}.$

So $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k(k+1)}$ converges to $1/2$.

4(a) $T_n = \{(b-a)/(2n)\} \cdot \{(y_0 + y_n) + 2(y_1 + y_2 + y_3 + \cdots + y_{n-1})\}$

$\therefore\ T_6 = \{(2.0 - 0.2)/12\} \cdot \{(3+1) + 2(3+5+4+2+3)\}$
$= \{(1.8)/12\}\{38\} = (.3)(38)/2 = (.3)(19) = 5.7$

(b) $S_n = \{(b-a)/3n\} \cdot \{(y_0 + y_n) + 4 y_{odd} + 2 y_{even}\}$
$= \{(2.0 - 0.2)/18\} \cdot \{(3+1) + 4(3+4+3) + 2(5+2)\}$
$= (1.8/18)(58) = (0.1)(58) = 5.8.$

5(a) If $f''(x)$ & $f''''(x)$ are continuous on $[a,b]$ and
$K_2 = \max\{|f''(x)|\}$ & $K_4 = \max\{|f''''(x)|\}$ for x in $[a,b]$
then $|E_T| \leq (b-a)^3 K_2/12n^2$ & $|E_S| \leq (b-a)^5 K_4/180n^4$.

(b) $\displaystyle\int \frac{x+2}{x^2-2x+5}\,dx = \int \frac{x+2}{(x-1)^2+4}\,dx = \int \frac{(u+1)+2}{u^2+4}\,du$

$\displaystyle = \int \frac{u}{u^2+4}\,du + \int \frac{3}{u^2+2^2}\,du$ $\boxed{\begin{array}{l}\text{Put } u = x-1. \text{ Then} \\ x = u+1 \ \&\ dx = du\end{array}}$

$\displaystyle = \frac{1}{2}\ln(u^2+4) + \frac{3}{2}\tan^{-1}\left(\frac{u}{2}\right) + C$

$\displaystyle = \frac{1}{2}\ln(x^2-2x+5) + \frac{3}{2}\tan^{-1}\left(\frac{x-1}{2}\right) + C = \ln\left(\sqrt{x^2-2x+5}\right) + \frac{3}{2}\tan^{-1}\left(\frac{x-1}{2}\right) + C.$

6(a) $a_2 = 1 + 6/a_1 = 1 + 6/4 = 5/2$, $a_3 = 1 + \frac{6}{a_2} = 1 + \frac{6}{5/2} = \frac{17}{5}$

$a_4 = 1 + 6/a_3 = 1 + 6/(17/5) = 47/17$.

(b) We know that $a_n = 1 + (6/a_{n-1})$. So

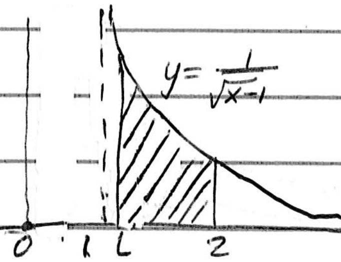
$\lim_{n\to\infty} a_n = 1 + 6/\lim_{n\to\infty} a_{n-1}$. $\therefore L = 1 + 6/L$

$\therefore L^2 = L + 6 \Rightarrow L^2 - L - 6 = 0 \Rightarrow (L+2)(L-3) = 0$

$\therefore L = -2$ or 3. Since $a_n \geq 0$ always, L must be 3.

7(a) $\int_1^2 \frac{1}{\sqrt{x-1}}\,dx = \lim_{L\to 1^+} \int_L^2 (x-1)^{-1/2}\,dx = \lim_{x\to 1^+} \left[\frac{(x-1)^{1/2}}{1/2}\right]_L^2$

$= \lim_{L\to 1^+} \{2 - 2(L-1)^{1/2}\} = 2 - 0 = 2$.

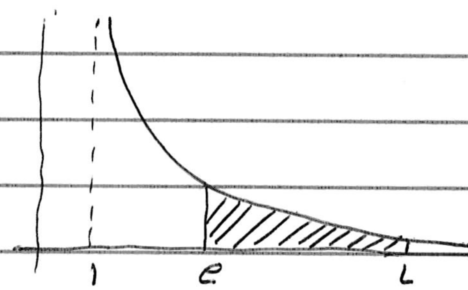


(b) $\int_e^{+\infty} \frac{1}{x(\ln x)^2}\,dx = \lim_{L\to\infty} \int_e^L \frac{1}{(\ln x)^2} \cdot \frac{dx}{x}$

$\boxed{\begin{array}{l}\text{Put } u = \ln x. \text{ Then } du = \frac{dx}{x} \\ x = e \Rightarrow u = 1,\ x = L \Rightarrow u = \ln L\end{array}}$

$= \lim_{L\to\infty} \int_1^{\ln L} \frac{1}{u^2}\,du = \lim_{L\to\infty} \left[\frac{-1}{u}\right]_1^{\ln L}$

$= \lim_{L\to\infty} 1 - \frac{1}{\ln L} = 1 - 0 = 1$.



8(a) $\int \cos^n x\,dx = \int \cos^{n-1} x \cdot \cos x\,dx$; $\boxed{\begin{array}{l}\text{Put } u = \cos^{n-1} x \ \&\ dv = \cos x\,dx \\ du = (n-1)\cos^{n-2} x(-\sin x)dx \ \&\ v = \sin x\end{array}}$

$= \int u\,dv = uv - \int v \cdot du$

$= \sin x \cdot \cos^{n-1} x - \int (n-1)\cos^{n-2} x \cdot (-\sin x) \cdot (\sin x)\,dx$

$= \sin x \cdot \cos^{n-1} x + (n-1)\int \cos^{n-2} x \cdot \sin^2 x\,dx$

$= \sin x \cdot \cos^{n-1} x + (n-1)\int \cos^{n-2} x \cdot (1 - \cos^2 x)\,dx$

$= \sin x \cdot \cos^{n-1} x + (n-1)\int \cos^{n-2} x \cdot dx - (n-1)\int \cos^n x \cdot dx$

$\therefore \{(n-1)+1\}\int \cos^n x\,dx = \sin x \cdot \cos^{n-1} x + (n-1)\int \cos^{n-2} x\,dx$

$\therefore \int \cos^n x\,dx = (\sin x \cdot \cos^{n-1} x)/n + \frac{(n-1)}{n}\int \cos^{n-2} x\,dx$

(b) $\int \cos^4 x\,dx = \frac{\sin x \cdot \cos^{4-1} x}{4} + \frac{3}{4}\int \cos^2 x\,dx$

$= \frac{\sin x \cos^3 x}{4} + \frac{3}{4}\left\{\frac{\sin x \cdot \cos^{2-1} x}{2} + \frac{1}{2}\int \underbrace{\cos^0 x}_{=1}\,dx\right\}$

$= \frac{\sin x \cdot \cos^3 x}{4} + \frac{3}{8}\sin x \cdot \cos x + \frac{3}{8}x + C$.