MAC 2313 - Calculus III — Florida Int'l Univ.

TEST #3 - Fall 2011 — TIME: 100 min.

Answer all 8 questions. An unjustified answer will receive little or no credit. No calculators or formula sheets are allowed. BEGIN EACH QUESTION ON A SEPARATE PAGE.

(12) 1(a) Let $\underset{\sim}{F} = \langle xz^2, x^2y, y^2z \rangle$. Find div($\underset{\sim}{F}$) & curl($\underset{\sim}{F}$).

(b) Suppose $x = u^2 w$, $y = v e^{-w}$, and $z = v e^{w}$. Find the Jacobian $\partial(x,y,z)/\partial(u,v,w)$.

(14) 2(a) Make a sketch of the solid region G that is above the xy plane, outside the cone $z = \sqrt{x^2+y^2}$, and inside the sphere $x^2+y^2+z^2 = 3$.

(b) Find the volume of G by using spherical coordinates

(10) 3. A thin wire, which is in the form of the helix

$$\underset{\sim}{r}(t) = \langle \sqrt{3}.t, \cos t, \sin t \rangle \qquad 0 \le t \le \pi,$$

has density $3y^2z$ at the point $\langle x, y, z \rangle$. Find the total mass of the wire by using a line integral.

(14) 4(a) Make a sketch of the portion S of the surface $z = (1-x^2-y^2)/2$ that lies above the xy-plane.

(b) If the density at the point $\langle x, y, (1-x^2-y^2)/2 \rangle$ is x^2+y^2, find the total mass of S by using a surface integral. [Hint: Use polar coordinates.]

PLEASE TURN OVER.

(12) 5. Using the theorems of Pappus, make a sketch and find

(a) the area of the surface formed by revolving the circle $(x-5)^2 + (y-2)^2 = 4$ about the y-axis.

(b) the volume of the solid formed by revolving the elliptical disk $\frac{(x-2)^2}{4} + \frac{y^2}{1} \leq 1$ about the y-axis.

(14) 6(a) Determine whether or not the vector field $\underset{\sim}{F} = \langle 2xyz - yz^2,\ x^2z - xz^2,\ x^2y - 2xyz \rangle$ is conservative.

(b) Let $\underset{\sim}{F} = \langle 3x^2 + 2e^{2x}\cos y,\ \ 2\sin y\cos y - e^{2x}\sin y \rangle$

Find all the scalar potential $\phi(x,y)$ such that $\underset{\sim}{F} = \text{grad}(\phi)$.

(10) 7. Let C be the parametric curve given by

$$\underset{\sim}{r}(t) = \langle \sin t,\ -\cos t,\ t^2 \rangle; \qquad 0 \leq t \leq \pi/2.$$

Also let $\underset{\sim}{F}$ be the vector field given by $\underset{\sim}{F} = \langle 6x, 3xy, 12\sqrt{z} \rangle$.

Find the work done by this vector field when a particle is moved from $\langle 0, -1, 0 \rangle$ to $\langle 1, 0, \pi^2/4 \rangle$ along C.

(14) 8(a) Make a sketch of the portion S of the surface $z = 1 - x^2 - y^2$ that lies above the xy plane; and orient it with unit normals that point outwards & upwards.

(b) Let $\underset{\sim}{F} = \langle 3x, 3y, z \rangle$. Find the total flux across the surface S in the vector field $\underset{\sim}{F}$.

END

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Solutions to Test #3 — Fall 2011

1(a) $\operatorname{div}(\underset{\sim}{F}) = \underset{\sim}{\nabla} \cdot \underset{\sim}{F} = \langle \partial/\partial x, \partial/\partial y, \partial/\partial z \rangle \cdot \langle x z^2, x^2 y, y^2 z \rangle$

$= \partial(x z^2)/\partial x + \partial(x^2 y)/\partial y + \partial(y^2 z)/\partial z = z^2 + x^2 + y^2.$

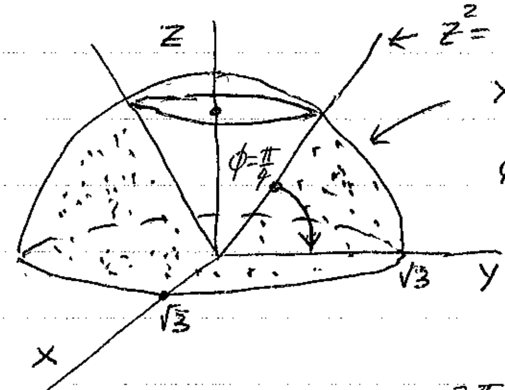
$\operatorname{curl}(\underset{\sim}{F}) = \underset{\sim}{\nabla} \times \underset{\sim}{F} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x z^2 & x^2 y & y^2 z \end{vmatrix} = \langle 2yz, 2xz, 2xy \rangle$

(b) $$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v & \partial x/\partial w \\ \partial y/\partial u & \partial y/\partial v & \partial y/\partial w \\ \partial z/\partial u & \partial z/\partial v & \partial z/\partial w \end{vmatrix} = \begin{vmatrix} 2uw & 0 & u^2 \\ 0 & e^{-w} & e^{w} \\ 0 & -ve^{-w} & ve^{w} \end{vmatrix}$$

$= (2uw)\left[e^{-w} \cdot ve^{w} - (-ve^{-w})e^{w}\right]$

$= (2uw)(v+v) = 4uvw.$

2(a)



$\leftarrow z^2 = x^2 + y^2$

$z^2 + z^2 = 3 \Rightarrow z^2 = 3/2$

$x^2 + y^2 + z^2 = 3 \Rightarrow \rho^2 = 3 \Rightarrow \rho = \sqrt{3}$

$\phi = \pi/4$ to $\phi = \pi/2$, $\quad \theta = 0$ to 2π

$\rho = 0$ to $\sqrt{3}$

(b) $$V = \iiint_G 1 \, dV = \int_{\theta=0}^{2\pi} \int_{\phi=\pi/4}^{\pi/2} \int_{\rho=0}^{\sqrt{3}} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \left(\int_0^{2\pi} 1 \, d\theta\right) \cdot \left(\int_{\pi/4}^{\pi/2} \sin\phi \, d\phi\right) \left(\int_0^{\sqrt{3}} \rho^2 \, d\rho\right)$$

$$= \left[\theta\right]_0^{2\pi} \cdot \left[-\cos\phi\right]_{\pi/4}^{\pi/2} \cdot \left[\frac{\rho^3}{3}\right]_0^{\sqrt{3}}$$

$$= [2\pi - 0] \cdot \left[0 - \left(-\sqrt{2}/2\right)\right] \cdot \left[\frac{(\sqrt{3})^3}{3} - 0\right]$$

$$= \cancel{2}\pi \cdot \frac{\sqrt{2}}{\cancel{2}} \cdot \frac{\cancel{3}\sqrt{3}}{\cancel{3}} = \pi \sqrt{2} \sqrt{3} = \pi\sqrt{6}.$$

<u>Note</u>: $z = \rho \sin\phi$. So when cone intersects sphere $\sqrt{3/2} = \sqrt{3} \sin\phi \Rightarrow 1/\sqrt{2} = \sin\phi \Rightarrow \phi = \pi/4$.

3. $\underset{\sim}{r}(t) = \langle \sqrt{3}.t \,,\ \cos t,\ \sin t \rangle \qquad 0 \le t \le \pi$

$\underset{\sim}{r}'(t) = \langle \sqrt{3}\,,\ -\sin t,\ \cos t \rangle$

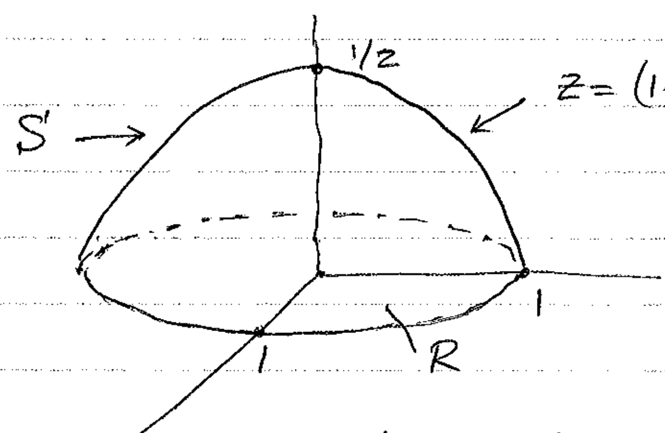
$M = \int_C \delta(\underset{\sim}{r})\, dl = \int_C 3y^2 z\, dl = \int_{t=0}^{\pi} 3y^2 z \left\|\frac{dr}{dt}\right\| dt$

$= \int_0^{\pi} 3y^2 z . \sqrt{3 + \sin^2 t + \cos^2 t}\ dt$

$= \int_0^{\pi} 3.\cos^2 t . \sin t .\ \sqrt{4}\ . dt$

$= 2 .\ \left[\cancel{3} . \left(-\frac{1}{\cancel{3}}\right) \cos^3(t) \right]_0^{\pi} = -2\left[(-1) - 1\right] = 4$

4(a)



$z = (1 - x^2 - y^2)/2$

$\underset{\sim}{r} = \langle x, y, \frac{1 - x^2 - y^2}{2} \rangle$

$R: \ 0 \le r \le 1\ ,\ 0 \le \theta \le 2\pi$

(b) $\frac{\partial r}{\partial x} \times \frac{\partial r}{\partial y} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -x \\ 0 & 1 & -y \end{vmatrix} = \langle x, y, 1 \rangle$

$M = \iint_S \delta(\underset{\sim}{r})\, dS = \iint_R (x^2 + y^2) . \left\| \frac{\partial r}{\partial x} \times \frac{\partial r}{\partial y} \right\| dA_{xy}$

$= \iint_R (x^2 + y^2) . \sqrt{x^2 + y^2 + 1}\ . dA_{xy}$

$= \int_0^{2\pi} \int_0^1 r^2 . \sqrt{r^2 + 1}\ . r\, dr\, d\theta$

$= \left[\int_0^{2\pi} d\theta\right] . \left[\int_{u=1}^{2} (u-1) . u^{1/2} . \frac{1}{2} du\right]$

$\boxed{\begin{aligned} &\text{Put } u = r^2 + 1 \\ &du = 2r\,dr \\ &r^2 = u - 1 \\ &r = 0 \Rightarrow u = 1 \\ &r = 1 \Rightarrow u = 2 \end{aligned}}$

$= [\theta]_0^{2\pi} . \frac{1}{2} \int_1^2 (u^{3/2} - u^{1/2})\, du$

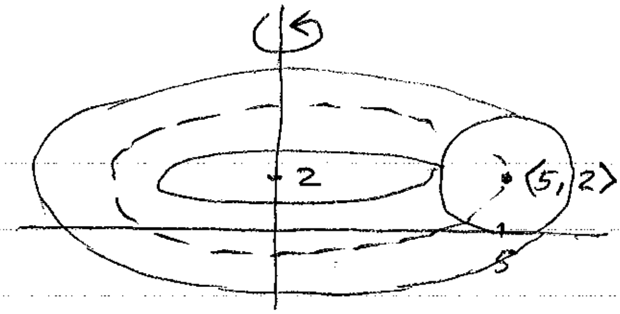
$= \cancel{2}\pi . \frac{1}{\cancel{2}} . \left[\frac{2u^{5/2}}{5} - \frac{2}{3} u^{3/2}\right]_1^2$

$= \pi . \left[\left(\frac{2.4\sqrt{2}}{15} - \frac{2.2\sqrt{2}}{3}\right) - \left(\frac{2}{5} - \frac{2}{3}\right)\right] = \frac{4\pi}{15}(\sqrt{2} - 1).$

5(a) Surface Area

$= \begin{pmatrix}\text{Length} \\ \text{of curve}\end{pmatrix} \cdot \begin{pmatrix}\text{Distance travelled} \\ \text{by its centroid}\end{pmatrix}$

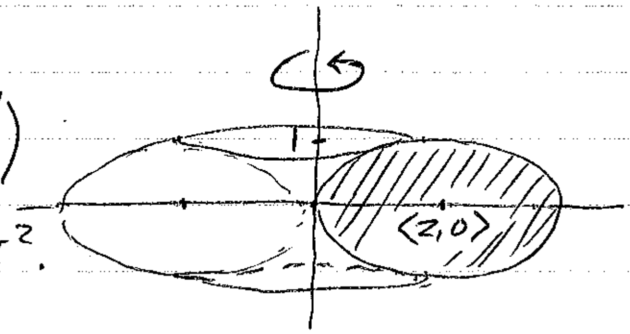
$= (2\pi \cdot 2) \cdot (2\pi \cdot 5) = 40\pi^2$



(b) Volume of solid

$= \begin{pmatrix}\text{Area of} \\ \text{disk}\end{pmatrix} \cdot \begin{pmatrix}\text{Dist. travelled} \\ \text{by its centroid}\end{pmatrix}$

$= (\pi \cdot 2 \cdot 1) \cdot [2\pi(2)] = 8\pi^2.$



6(a) $\text{curl}(\underset{\sim}{F}) = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2xyz - yz^2 & x^2z - xz^2 & x^2y - 2xyz \end{vmatrix}$

$= \langle (x^2 - 2xz) - (x^2 - 2xz),\ (2xy - 2yz) - (2xy - 2yz),$
$(2xz - z^2) - (2xz - z^2) \rangle$

$= \langle 0, 0, 0 \rangle = \underset{\sim}{0}.$

$\therefore \underset{\sim}{F}$ is a conservative vector field.

(b) Suppose $\nabla\phi = \underset{\sim}{F}$. Then $\left\langle \frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y} \right\rangle = \langle F_1, F_2 \rangle$

$\therefore \frac{\partial\phi}{\partial x} = 3x^2 + 2e^{2x}\cos(y) \Rightarrow \phi = x^3 + e^{2x}\cos y + g(y)$

and $\frac{\partial\phi}{\partial y} = 2\sin y \cos y - e^{2x}\sin y.$

But $\frac{\partial\phi}{\partial y} = 0 + e^{2x} \cdot (-\sin y) + g'(y)$ also

Hence $g'(y) = 2\sin y \cos y. \quad \therefore g(y) = \sin^2(y) + C$

Thus

$$\phi(x,y) = x^3 + e^{2x} \cdot \cos(y) + g(y)$$
$$= x^3 + e^{2x} \cdot \cos(y) + \sin^2(y) + C.$$

7(a) $\underset{\sim}{r}(t) = \langle \sin t, -\cos t, t^2 \rangle \qquad 0 \le t \le \pi/2$

$\underset{\sim}{r}'(t) = \langle \cos t, \sin t, 2t \rangle$

Work done $= \int_C \underset{\sim}{F} \cdot d\underset{\sim}{r} = \int_{t=0}^{\pi/2} \underset{\sim}{F}(\underset{\sim}{r}(t)) \cdot \left(\frac{dr}{dt}\right) dt$

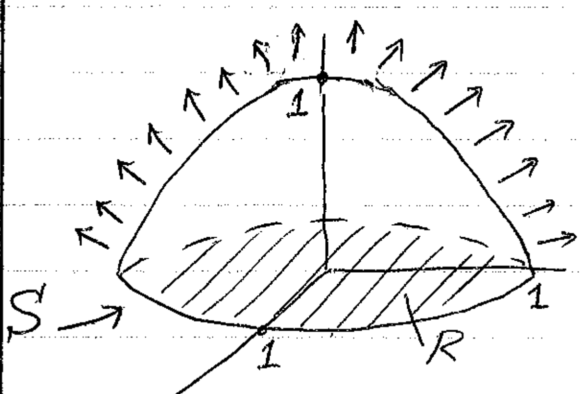
$= \int_{t=0}^{\pi/2} \langle 6x, 3xy, 12\sqrt{z} \rangle \cdot \langle \cos t, \sin t, 2t \rangle \, dt$

$= \int_0^{\pi/2} (6x \cos t + 3xy \sin t + 24t\sqrt{z}) \, dt$

$= \int_0^{\pi/2} (6 \sin t \cos t - 3 \sin^2 t \cos t + 24 t^2) \, dt$

$= \left[\cancel{6}^{3} \cdot \frac{1}{\cancel{2}} \sin^2 t - \cancel{3} \cdot \frac{1}{\cancel{3}} \sin^3 t + \cancel{24}^{8} \frac{t^3}{\cancel{3}}\right]_0^{\pi/2} = 2 + \pi^3.$

8(a)



$z = 1 - x^2 - y^2$

$\underset{\sim}{r} = \langle x, y, 1 - x^2 - y^2 \rangle$

$\underset{\sim}{F} = \langle 3x, 3y, z \rangle$

Use x & y to parameterise S

(b) $\frac{\partial r}{\partial x} \times \frac{\partial r}{\partial y} = \begin{vmatrix} i & j & k \\ 1 & 0 & -2x \\ 0 & 1 & -2y \end{vmatrix} = \langle 2x, 2y, 1 \rangle$

Flux across S $= \iint_S \underset{\sim}{F} \cdot d\underset{\sim}{S} = \iint_R \underset{\sim}{F} \cdot \left(\frac{\partial r}{\partial x} \times \frac{\partial r}{\partial y}\right) dA_{xy}$

$= \iint_R (\langle 3x, 3y, z \rangle \cdot \langle 2x, 2y, 1 \rangle) \, dA_{xy}$

$= \iint_R (6x^2 + 6y^2 + z) \, dA_{xy} = \iint_R (6x^2 + 6y^2 + 1 - x^2 - y^2) dA$

$= \iint_R (5x^2 + 5y^2 + 1) \, dA_{xy} = \int_{\theta=0}^{2\pi} \int_{r=0}^{1} (5r^2 + 1) \cdot r \, dr \, d\theta$

$= \left(\int_0^{2\pi} d\theta\right) \cdot \int_0^1 (5r^3 + r) \, dr$

$= [\theta]_0^{2\pi} \cdot \left[5\frac{r^4}{4} + \frac{r^2}{2}\right]_0^1$

$= 2\pi \cdot \left(\frac{5}{4} + \frac{1}{2}\right) = 2\pi \cdot \frac{7}{4} = \frac{7\pi}{2}.$