

Answer all 6 questions. No calculators, cell-phones, or class-notes are allowed. An un-justified answer will receive little or no credit. Begin each of the 6 questions on 6 separate pages.

- (16) 1.(a) How many **integers** in $\{1, 2, 3, \dots, 300\}$ are *divisible* by **neither** 10 **nor** 14?
 (b) Find the **permutation** which has $\langle 4, 0, 1, 2, 1, 0 \rangle$ as its *inversion sequence* and check that your answer is correct.
- (16) 2.(a) Write down what is the **Multinomial Theorem** in full details. Then use it to find the **coefficient** of $x^4 y^6 z^3$ in the expansion of $(3x + 2y^3 - xz)^6$.
 (b) Find the number of **integer solutions** of the equation: $x_1 + x_2 + x_3 = 8$ with $x_1 \geq -2$, $x_2 \geq -3$, and $x_3 \geq 4$.
- (16) 3.(a) Write down what is the **Second version** of the **Inclusion-Exclusion Theorem**.
 (b) Find the number of **15-combinations** of the multi-set $F = [16.a, 5.b, 3.c]$.
 [You should leave your answer in terms of simplified Binomial coefficients.]
- (16) 4. (a) How many permutations of $\{2, 3, 4, 5, 6, 7, 8, 9\}$ have *each even number* going to an **even number** and no **odd number** going to *itself*?
 (b) How many permutations of $\{3, 4, 5, 6, 7, 8\}$ have *no even number* going to *itself*? [You may leave your answer in terms of simplified factorials.]
- (16) 5.(a) Write down what is **Pascal's identity** and give a **combinatorial proof** of it.
 (b) Prove that for any **non-negative integers** k & n with $0 \leq k \leq n$, we have:

$$\binom{k}{k} + \binom{k+1}{k} + \binom{k+2}{k} + \dots + \binom{n}{k} = \binom{n+1}{k+1}.$$
- (20) 6.(a) Define the **positive & ultimate sets** w.r.t the subsets $A_1, A_2, A_3, \dots, A_n$ of U .
 (b) Let $n \geq 2$. Find the **value** of each of the following sums (in simplified form).
 (i) $2 \cdot \binom{n}{0} - 3 \cdot \binom{n}{1} + 4 \cdot \binom{n}{2} - 5 \cdot \binom{n}{3} + \dots + (-1)^n \cdot (n+2) \cdot \binom{n}{n} = (?)$
 (ii) $\frac{1}{2} \cdot \binom{n}{0} + \frac{1}{4} \cdot \binom{n}{1} + \frac{1}{6} \cdot \binom{n}{2} + \dots + \frac{1}{2(n+1)} \cdot \binom{n}{n} = (??) \quad \text{END.}$

Solutions to Test #1

Fall 2026

#1(a) Let $U = \{1, 2, 3, \dots, 300\}$ & $A_k = \{x \in U : x \text{ is divisible by } k\}$. Then

$$\text{Answer} = |A_{10}^c \cap A_{14}^c| = |U| - |A_{10}| - |A_{14}| + |A_{10} \cap A_{14}| = 300 - \left\lfloor \frac{300}{10} \right\rfloor - \left\lfloor \frac{300}{14} \right\rfloor + \left\lfloor \frac{300}{\text{lcm}(10, 14)} \right\rfloor$$

$$= 300 - 30 - 21 + \left\lfloor \frac{300}{70} \right\rfloor = 249 + 4 = \boxed{253}$$

(b) Write down $\langle 6 \rangle$, then $\langle 6, 5 \rangle$ bec. $i_5 = 1$, then $\langle 6, 5, 4 \rangle$ bec. $i_4 = 2$, then $\langle 6, 3, 5, 4 \rangle$ bec. $i_3 = 1$, then $\langle 2, 6, 3, 5, 4 \rangle$ bec. $i_2 = 0$, then $\langle 2, 6, 3, 5, 1, 4 \rangle$ bec. $i_1 = 4$. Checking $i_1 = 4, i_2 = 0, i_3 = 1, i_4 = 2, i_5 = 1$ & $i_6 = 0$. yes!

#2(a) For any $k, n \in \mathbb{N}$, $(x_1 + \dots + x_k)^n = \sum_{\{n_1, \dots, n_k\} : n_1 + n_2 + \dots + n_k = n} \binom{n}{n_1, n_2, \dots, n_k} x_1^{n_1} x_2^{n_2} \dots x_k^{n_k}$, where $\binom{n}{n_1, n_2, \dots, n_k} = \frac{n!}{n_1! n_2! \dots n_k!}$

Coeff. of $x^4 y^6 z^3$ in the expansion of $(3x + 2y^3 - xz)^6$

$$= \text{coeff. of only the term from } (3x)^1 (2y^3)^2 (-xz)^3$$

$$= \text{coeff. of } \binom{6}{1, 2, 3} (3x)^1 (2y^3)^2 (-xz)^3 = \frac{6!}{1! 2! 3!} (3)^1 (2)^2 (-1)^3$$

$$= -60(12) = \boxed{-720}$$

(b) Put $x_1 = y_1 - 2$, $x_2 = y_2 - 3$, and $x_3 = y_3 + 4$. Then

$$x_1 + x_2 + x_3 = 8 \text{ becomes } (y_1 - 2) + (y_2 - 3) + (y_3 + 4) = 8 \text{ \& } y_1 + y_2 + y_3 = 9$$

So answer = no. of non-neg. integer solutions of $y_1 + y_2 + y_3 = 9$ which is $\binom{9+3-1}{3-1} = \binom{11}{2} = \frac{11(10) \cdot 9!}{2! \cdot 9!} = \frac{110}{2} = \boxed{55}$

#3(a) The 2nd version of the Inclusion-Exclusion Principle says

$$|A_1 \cup A_2 \cup \dots \cup A_n| = \sum_{k=1}^n (-1)^{k-1} \left\{ \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} |A_{i_1} \cap A_{i_2} \cap \dots \cap A_{i_k}| \right\} \text{ where } A_1, \dots, A_n \text{ are subsets of } U$$

(b) Let $M = [\infty, a, \infty, b, \infty, c]$ & U = set of all 15-comb. of M . Then put

A = set of all 15-comb. of M with $\geq 17a$'s, B = set of all 15-comb. of M with $\geq 6b$'s, & C = set of all 15-comb. of M with $\geq 4c$'s. Then $A \cap B = A \cap C = A \cap B \cap C$, and $B \cap C$ = set of all 15-comb. of M with $\geq 6b$'s & $\geq 4c$'s. Hence

$|B|$ = |set of all 9-comb. of M |, $|C|$ = |set of all 11-comb. of M | & $|B \cap C|$ = |set of all 5-comb. of M |. So no. of 15-comb. of $F = [17, a, 6, b, 3, c]$ will be

$$|A^c \cap B^c \cap C^c| = |U| - |A| - |B| - |C| + |A \cap B| + |A \cap C| + |B \cap C| - |A \cap B \cap C|$$

$$= \binom{15+3-1}{3-1} - 0 - \binom{9+3-1}{3-1} - \binom{11+3-1}{3-1} + 0 + 0 + \binom{5+3-1}{3-1} = \binom{17}{2} - \binom{11}{2} - \binom{13}{2} + \binom{7}{2}$$

#4(a) Let $D = \{3, 5, 7, 9\}$ & $E = \{2, 4, 6, 8\}$. Then there are $4!$ ways of permuting the elements of E & $D_4 = 9$ ways of deranging the elements of D . So no. of permutations of $\{2, 3, 4, 5, 6, 7, 8, 9\}$ with even $\rightarrow$ even & no odd no. going to itself will be $4! \cdot D_4 = 24(9) = \boxed{216}$.

(b) Let $B = \{4, 6, 8\}$. Put A_i = set of permutations of $\{3, 4, 5, 6, 7, 8\}$ with $i \rightarrow i$, and U = set of all permutation of $\{3, 4, 5, 6, 7, 8\}$. Then answer = $|A_4^c \cap A_6^c \cap A_8^c| = |U| - |A_4| - |A_6| - |A_8| + |A_4 \cap A_6| + |A_4 \cap A_8| + |A_6 \cap A_8| - |A_4 \cap A_6 \cap A_8| = 6! - 3(5!) + 3(4!) - (3!)$.

#5(a) Pascal's identity is: $\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}$ for $0 \leq k \leq n$, $n, k \in \mathbb{N}$.

Now $\binom{n+1}{k+1}$ = no. of subsets of $\{1, 2, 3, \dots, n+1\}$ with $(k+1)$ elements
 = no. of $(k+1)$ -subsets of $\{1, 2, 3, \dots, n+1\}$ which contains "1"
 + no. of $(k+1)$ -subsets of $\{1, 2, 3, \dots, n+1\}$ which don't contain "1"
 = no. of k -subsets of $\{2, 3, \dots, n+1\}$ with "1" added
 + no. of k -subsets of $\{2, 3, \dots, n+1\}$ with "1" excluded
 = $\binom{n}{k} + \binom{n}{k+1}$. We have thus proved Pascal's identity.

(b) We will prove the result by parametric induction on n . (k will be the temporarily fixed parameter).

Basis: If $n=k$, then we have $\binom{k}{k} = 1 = \binom{k+1}{k+1} = \binom{n+1}{k+1}$. So the result is true for $n=k$.

Ind. Step: Now suppose the result is true for n (where $n \geq k$). Then $\binom{k}{k} + \binom{k+1}{k} + \binom{k+2}{k} + \dots + \binom{n}{k} = \binom{n+1}{k+1}$.

$$\begin{aligned} \text{So } \binom{k}{k} + \binom{k+1}{k} + \binom{k+2}{k} + \dots + \binom{n}{k} + \binom{n+1}{k} &= \binom{n+1}{k+1} + \binom{n+1}{k} \\ &= \binom{(n+1)+1}{k+1}. \end{aligned}$$

by Pascal's identity.

So if the result is true for n , then it will be true for $n+1$.

By the Principle of Parametric Induction, it follows that the result is true for all $n \geq k$.

#6(a) Let $A_1, A_2, A_3, \dots, A_n$ be subsets of a universal set U . A positive set w.r.t U & $A_1, A_2, \dots, A_n$ is any set of the form $U \cap A_{i_1} \cap A_{i_2} \cap \dots \cap A_{i_k}$ where $\langle i_1, i_2, \dots, i_k \rangle$ is any subsequence of $\langle 1, 2, 3, \dots, n \rangle$. An ultimate set w.r.t U & $A_1, A_2, \dots, A_n$ is any set of the form $X_1 \cap X_2 \cap \dots \cap X_n$ where $X_i = A_i$ or A_i^c , for each $i = 1, 2, 3, \dots, n$.

(b) (i) We know $\binom{n}{0} \cdot x^0 + \binom{n}{1} \cdot x^1 + \binom{n}{2} \cdot x^2 + \dots + \binom{n}{n} \cdot x^n = (1+x)^n$ by the Binomial Theorem. So multiplying by x^2 on both sides

$$\binom{n}{0} \cdot x^2 + \binom{n}{1} x^3 + \binom{n}{2} x^4 + \dots + \binom{n}{n} x^{n+2} = x^2 \cdot (1+x)^n$$

Differentiating both sides gives

$$2 \cdot \binom{n}{0} x^1 + 3 \cdot \binom{n}{1} x^2 + 4 \cdot \binom{n}{2} x^3 + \dots + (n+2) \binom{n}{n} x^{n+1} = 2x \cdot (1+x)^n + x^2 \cdot n \cdot (1+x)^{n-1}$$

Putting $x = -1$ gives

$$-2 \binom{n}{0} + 3 \binom{n}{1} - 4 \binom{n}{2} + \dots + (-1)^{n+1} \cdot (n+2) \binom{n}{n} = 0 + 0 = 0$$

Dividing both sides by (-1) gives $2 \binom{n}{0} - 3 \binom{n}{1} + 4 \binom{n}{2} - 5 \binom{n}{3} + \dots + (-1)^n \cdot (n+2) \binom{n}{n} = \boxed{0}$

(ii) We know: $\binom{n}{0} \cdot x^0 + \binom{n}{1} x^1 + \binom{n}{2} x^2 + \dots + \binom{n}{n} \cdot x^n = (1+x)^n$

Integrating both sides from 0 to 1, we get

$$\int_0^1 \left\{ \binom{n}{0} \cdot x^0 + \binom{n}{1} x^1 + \dots + \binom{n}{n} \cdot x^n \right\} dx = \int_0^1 (1+x)^n dx$$

$$\therefore \left[\binom{n}{0} \frac{x^1}{1} + \binom{n}{1} \frac{x^2}{2} + \binom{n}{2} \frac{x^3}{3} + \dots + \binom{n}{n} \frac{x^{n+1}}{n+1} \right]_0^1 = \left[\frac{(1+x)^{n+1}}{n+1} \right]_0^1$$

because $\int (1+x)^n dx = \int u^n \cdot du = \frac{u^{n+1}}{n+1} + C \leftarrow \begin{array}{l} \text{Put } u=1+x \\ \text{then } du=dx \end{array}$

$$= \frac{(1+x)^{n+1}}{n+1} + C$$

So $\left[\frac{1}{1} \binom{n}{0} + \frac{1}{2} \binom{n}{1} + \frac{1}{3} \binom{n}{2} + \dots + \frac{1}{n+1} \binom{n}{n} \right] - [0] = \frac{2^{n+1} - 1}{n+1}$

Dividing both sides by 2 gives

$$\therefore \frac{1}{2} \binom{n}{0} + \frac{1}{4} \binom{n}{1} + \frac{1}{6} \binom{n}{2} + \dots + \frac{1}{2(n+1)} \binom{n}{n} = \boxed{\frac{2^{n+1} - 1}{2(n+1)}}$$

END