MAD 4203 - INTROD TO COMBINATORICS — FLORIDA INT'L UNIV.

TEST #2 - FALL 2017 — TIME: 75 min.

Answer all 6 questions. ***No notes, calculators or cellphones are allowed.*** *An* ***unjustified*** *answer will receive little or no credit. BEGIN EACH OF THE 6 QUESTIONS ON 6 SEPARATE PAGES.*

(15) 1. Find *the solution* of the recurrence equation $x_{n+2} - 2x_{n+1} - 8x_n = 0$ with the *initial conditions* $x_0 = 5$ and $x_1 = 2$.

(20) 2. Find the *complementary solution* and give (with reasons) the *minimum form* of a *particular solution* of each of the following *recurrence equations.*

(a) $(E^2 + E - 2I)(E^2 + I)(x_n) = 4n^2$.

(b) $(E^2 - 2E + 2I)(E^2 - 4E + 4I)(x_n) = 5n.(2)^n$.

(20) 3(a) Use the *method of generating functions* to find the solution of the recurrence equation: $a_n + 3a_{n-1} - 4 = 0$ with $a_0 = 3$.

(b) Let $M = [\infty.a, \infty.b, 9.c]$. Find the *generating function* of the set of all *n-combinations* of M with at least four a's, an even numbers of b's, and an odd numbers of c's. [*Simplify your answer as far as possible.*]

(15) 4(a) Define what are the operators E and Δ that operates on a sequence $\langle x_n \rangle$.

(b) Let $h_k = 3k^2 - 3k + 1$. Find the *zero-column* of $\langle h_k \rangle_{k \in \mathbb{N}}$ & use it to get a formula for the sum $h_0 + h_1 + h_2 + \ldots + h_n$. [*Simplify your answer as far as possible.*]

(15) 5. (a) Write down what is the *function form* of the Pigeon-Hole Principle.

(b) Let $\langle a_1, a_2, a_3, \ldots, a_n \rangle$ be any sequence of n integers. Prove that there is a non-empty segment of consecutive terms of this sequence whose sum is a multiple of n.

(15) 6(a) Define what are the *Stirling coefficients of the First & Second kinds.*

(b) Prove for any k with $0<k<p$, the Stirling coefficientss of the *first kind* satisfy

$$\begin{bmatrix} p \\ k \end{bmatrix} = \begin{bmatrix} p-1 \\ k-1 \end{bmatrix} + (p-1).\begin{bmatrix} p-1 \\ k \end{bmatrix}.$$

MAD 4203 – Combinatorics — Florida Int'l Univ.

Solutions to Test #2 — Fall 2017

1. $x_{n+2} - 2x_{n+1} - 8x_n = 0$ with $x_0 = 5$ and $x_1 = 2$. This becomes $(E^2 - 2E - 8I)x_n = 0 \Rightarrow (E+2I)(E-4I)x_n = 0$. So $E = -2$ or 4

$\therefore\ x_n = A.(-2)^n + B.(4)^n$. Now $x_0 = 5 \Rightarrow 5 = A + B$

So $B = 5 - A$. $\quad x_1 = 2 \Rightarrow 2 = -2A + 4B$

$\therefore\ 2 = -2A + 4(5-A) \Rightarrow 6A = 18 \Rightarrow A = 3$. $\therefore\ B = 5 - 3 = 2$.

So $x_n = 3.(-2)^n + 2.(4)^n$. Check: $x_0 = 3(-2)^0 + 2(4)^0 = 5$ ✓

$x_1 = 3(-2)^1 + 2(4)^1 = 2$ ✓

2.(a) $(E^2 + E - 2I)(E^2 + I)x_n = 0 \Rightarrow (E-I)(E+2I)(E-iI)(E+iI)x_n = 0$.

$\therefore\ E = 1, -2, i$, or $-i$. So complementary solution is given by $x_n^c = A.(1)^n + B.(-2)^n + C.(i)^n + D.(-i)^n$

$(E-I)(E+2I)(E-iI)(E+iI)x_n = 4.n^2 = 4n^2.(1)^n$. Since $4n^2$ is a polynomial of degree 2 and 1 is a root of the auxiliary equation of multiplicity 1, the minimal form of a particular solution will be

$x_n^p = (a + bn + cn^2).n^1.(1)^n = an + bn^2 + cn^3$.

(b) $(E^2 - 2E + 2I)(E^2 - 4E + 4I)x_n = 0 \Rightarrow (E^2 - 2E + 2I)(E - 2I)^2 x_n = 0$.

$E^2 - 2E + 2I = 0 \Rightarrow E = \left[-(-2) \pm \sqrt{(-2)^2 - 4(1)(2)}\right]/2 = 1 \pm i$.

So the roots of the auxiliary equations are $2, 2, 1+i, 1-i$.

$\therefore\ x_n^c = A.(2)^n + Bn.(2)^n + C.(1+i)^n + D.(1-i)^n$.

$(E^2 - 2E + 2I)(E - 2I)^2 x_n = 5n.(2)^n$. Since $5n$ is a polynomial of degree 1 and 2 is a root of the auxiliary equation of multiplicity 2, the minimal form of a particular solution will be

$x_n^p = (a + bn).n^2.(2)^n = an^2.(2)^n + bn^3.(2)^n$

Notice in both (a) & (b) x_n^c & x_n^p have no terms in common. [A term is an expr. like $Bn.(2)^n$ or $an^2.(2)^n$.]

3(a) Let $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$.

Then $3x.f(x) = 3a_0 x + 3a_1 x^2 + \cdots + 3a_{n-1} x^n + \cdots$

also $-4/(1-x) = -4 - 4x - 4x^2 - \cdots - 4x^n - \cdots$

$\therefore (1+3x)f(x) - 4/(1-x) = (a_0-4) + (a_1+3a_0-4)x + \cdots + (a_n+3a_{n-1}-4)x^n + \cdots$

$= (3-4) + 0.x + 0.x^2 + \cdots + 0.x^n + \cdots = -1$

$\therefore (1+3x)f(x) = -1 + 4/(1-x) = (3+x)/(1-x)$.

$\therefore f(x) = \dfrac{3+x}{(1-x)(1+3x)} = \dfrac{A}{1-x} + \dfrac{B}{1+3x}$. $\therefore A(1+3x) + B(1-x) = 3+x$

$\therefore A+B = 3$ and $3A - B = 1$. So $B = 3A-1 \Rightarrow A+3A-1 = 3 \Rightarrow 4A = 4$.
So $A=1$ & $B=2$.

$\therefore f(x) = \dfrac{1}{1-x} + \dfrac{3}{1-(-3x)} = 1.[1 + x + x^2 + \cdots + x^n + \cdots]$
$+2[1 + (-3x) + (-3x)^2 + \cdots + (-3x)^n + \cdots]$

$\therefore a_n =$ coeff. of x^n in the exp. of $f(x) = 1 + 2(-3)^n$. Check $a_0 = 1+2 = 3$.

(b) see end of solutions — after 6(b).

4(a) The operator E is defined by $E(\langle x_n \rangle_{n\in\mathbb{N}}) = \langle x_{n+1} \rangle_{n\in\mathbb{N}}$.
The operator Δ is defined by $\Delta(\langle x_n \rangle_{n\in\mathbb{N}}) = \langle x_{n+1} - x_n \rangle_{n\in\mathbb{N}}$.
Note that $\Delta = (E - I)$.

(b)

$\langle k \rangle_{k\in\mathbb{N}}$	0	1	2	3	4	5
$\langle h_k \rangle$	1	1	7	19	37	61
$\langle \Delta^1 h_k \rangle$	0	6	12	18	24	.
$\langle \Delta^2 h_k \rangle$	6	6	6	6	.	.
$\langle \Delta^3 h_k \rangle$	0	0	0	.	.	.

So zero-column of $\langle h_k \rangle$ is $\langle 1, 0, 6, 0, 0, \cdots \rangle$

Thus $h_k = 1.\binom{k}{0} + 0.\binom{k}{1} + 6.\binom{k}{2}$. Hence

$\sum_{k=0}^{n} h_k = 1.\sum_{k=0}^{n}\binom{k}{0} + 0.\sum_{k=0}^{n}\binom{k}{1} + 6.\sum_{k=0}^{n}\binom{k}{2} = 1.\binom{n+1}{1} + 0.\binom{n+1}{2} + 6.\binom{n+1}{3}$

$= (n+1)/1! + 0 + 6(n+1)(n)(n-1)/3! = (n+1)[n^2 - n + 1] = n^3 + 1$.

5(a) <u>PHP function form</u>: If $f: P \to H$ is a function and $|P| > |H|$, then there exists $x_1, x_2 \in P$ with $x_1 \neq x_2$ such that $f(x_2) = f(x_1)$.

(b) Let $H = \{0, 1, 2, \ldots, n-1\}$ and $P = \{0, a_1, a_1+a_2, \ldots, a_1+a_2+\cdots+a_n\}$.
Then $|P| = n+1 > n = |H|$. Now define $f: P \to H$ by $f(x) = x \pmod{n}$
Since $|P| > |H|$, there exists two elements $x_1 = a_1 + a_2 + \cdots + a_i$ and $x_2 = a_1 + a_2 + \cdots + a_i + a_{i+1} + \cdots + a_j$ with $0 \le i < j \le n$ such that

5(b) $f(x_2) = f(x_1)$. Note when $i=0$, $a_1 + \cdots + a_i = \sum_{i \in \emptyset} a_i = 0$.
Thus $a_1 + \cdots + a_i + a_{i+1} + \cdots + a_j \equiv a_1 + \cdots + a_j \pmod{n}$
Hence $a_{i+1} + a_{i+2} + \cdots + a_j \equiv 0 \pmod{n}$. So the sum of the terms in the segment $\langle a_{i+1}, a_{i+2}, \ldots, a_{j-1}, a_j \rangle$ will be an integer multiple of n. Since $i<j$, this segment will be non-empty. Note when $i=0$, $\langle a_{i+1}, \ldots, a_j \rangle = \langle a_1, a_2, \ldots, a_j \rangle$ and when $i=j-1$, $\langle a_{i+1}, a_{i+2}, \ldots, a_j \rangle = \langle a_{i+1} \rangle = \langle a_j \rangle$.

6(a) The Stirling coefficients of the first kind are defined to be the unique integers $\left[{p \atop k}\right]$ such that $[n]_p = \sum_{k=0}^{p} (-1)^{p-k} \cdot \left[{p \atop k}\right] \cdot n^k$. The Stirling coefficients of the Second kind are define to be the unique integers $\left\{{p \atop k}\right\}$ such that $n^p = \sum_{k=0}^{p} \left\{{p \atop k}\right\} \cdot [n]_k$. Here $[n]_p = (n-0)(n-1)(n-2)\cdots(n-(p-1))$.

(b) We know that $[n]_p = (-1)^{p-p}\left[{p \atop p}\right] n^p + \sum_{k=1}^{p-1} (-1)^{p-k} \cdot \left[{p \atop k}\right] \cdot n^k + (-1)^{p-0}\left[{p \atop 0}\right] \cdot n^0$ (*)

So $[n]_p = [n]_{p-1} \cdot [n-(p-1)] = \sum_{k=0}^{p-1} (-1)^{(p-1)-k} \left[{p-1 \atop k}\right] \cdot n^k \cdot [n-(p-1)]$

$$= \sum_{k=0}^{p-1} (-1)^{p-(k+1)} \left[{p-1 \atop k}\right] n^{k+1} - (p-1) \cdot \sum_{k=0}^{p-1} (-1)^{(p-k)-1}$$

$\boxed{\text{Put } i=k+1 \text{ The } k=i-1}$

$$= \sum_{i=1}^{p} (-1)^{p-i} \left[{p-1 \atop i-1}\right] n^i + (p-1) \cdot \sum_{k=0}^{p-1} (-1)^{p-k}$$

$$= \sum_{k=1}^{p} (-1)^{p-k} \left[{p-1 \atop k-1}\right] n^k + \sum_{k=1}^{p-1} (-1)^{p-k} \cdot (p-1) \cdot \left[{p-1 \atop k}\right] n^k + (p-1) \cdot (-1)^{p-0} \left[{p-1 \atop 0}\right] n^0$$

$$= (-1)^{p-p} \left[{p-1 \atop p-1}\right] \cdot n^p + \sum_{k=1}^{p} (-1)^{p-1} \left\{ \left[{p-1 \atop k-1}\right] + (p-1)\left[{p-1 \atop k}\right] + (p-1) \cdot (-1) \cdot \left[{p-1 \atop 0}\right] \right.$$

Equating the coefficients of (*) with the last equation, we get $\left[{p \atop k}\right] = \left[{p-1 \atop k-1}\right] + (p-1) \cdot \left[{p-1 \atop k}\right]$ for $1 \le k \le p-1$.

Oops! I forgot to do Problem # 3(b).

3(b) Gen. func. $= (x^4 + x^5 + x^6 + \cdots)(x^0 + x^2 + x^4 + \cdots)(x^1 + x^3 + \cdots + x^9)$

$= x^4(1 + x + x^2 + \cdots) \cdot (1 + x^2 + x^4 + \cdots) \cdot x(1 + x^2 + x^4 + x^6 + x^8)$

$= x^5 \cdot \frac{1}{1-x} \cdot \frac{1}{1-x^2} \cdot \frac{1-x^{10}}{1-x^2} = \frac{x^5(1-x^{10})}{(1-x)^3(1+x)^2}$.