§1. Two forms of the Inclusion-Exclusion Principle

Ex. 1. How many integers in the set \( U = \{1, 2, 3, \ldots, 1000\} \) are divisible by 3, 5, or 7.

Sol. Let \( A = \{ n \in U : n \text{ is divisible by } 3 \} \), \( B = \{ n \in U : n \text{ is divisible by } 5 \} \), and \( C = \{ n \in U : n \text{ is divisible by } 7 \} \).

Then the answer to our problem is

\[
|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|
\]

Now \( A = \{ 3(1), 3(2), 3(3), \ldots, 3(k) \} \) where \( k \) is the largest integer \( \leq 1000/3 \). So

\[
|A| = \left\lfloor \frac{1000}{3} \right\rfloor. \text{ Similarly } |B| = \left\lfloor \frac{1000}{5} \right\rfloor \text{ and } |C| = \left\lfloor \frac{1000}{7} \right\rfloor.
\]

Also \( A \cap B = \{ n \in U : n \text{ is divisible by both } 3 \text{ and } 5 \} = \{ n \in U : n \text{ is divisible by } \text{lcm}(3, 5) \} = \{ n \in U : n \text{ is divisible by } 15 \} \). So

\[
|A \cap B| = \left\lfloor \frac{1000}{15} \right\rfloor. \text{ Similarly } |A \cap C| = \left\lfloor \frac{1000}{3} \right\rfloor \text{ and } |B \cap C| = \left\lfloor \frac{1000}{7} \right\rfloor.
\]

and \( |A \cap B \cap C| = \left\lfloor \frac{1000}{3 \times 5 \times 7} \right\rfloor. \text{ Thus }

\[
|A \cup B \cup C| = \left\lfloor \frac{1000}{3} \right\rfloor + \left\lfloor \frac{1000}{5} \right\rfloor + \left\lfloor \frac{1000}{7} \right\rfloor - \left\lfloor \frac{1000}{3 \times 5} \right\rfloor - \left\lfloor \frac{1000}{3 \times 7} \right\rfloor - \left\lfloor \frac{1000}{5 \times 7} \right\rfloor + \left\lfloor \frac{1000}{3 \times 5 \times 7} \right\rfloor.
\]

\[
= 333 + 200 + 142 - 66 - 42 - 28 + 9 = 543.
\]

Ex. 2. Prove \( n! = 1 + \sum_{k=0}^{n} k \cdot (k!) \) by (a) induction on \( n \) for \( n \geq 0 \).

(b) combinatorially.
Def: let $A_1, A_2, \ldots, A_n$ be subsets of a universal set $U$. A positive set w.r.t. $U$ & $A_1, A_2, \ldots, A_n$ is any set of the form $\bigcap A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_k}$ where $(i_1, i_2, \ldots, i_k)$ is any subsequence (including the empty subsequence $\langle \rangle$) of the sequence $(1, 2, 3, \ldots, n)$.

We usually leave out the $U$ when $(i_1, \ldots, i_k)$ is not the empty sequence - and we also leave out the intersection signs. So we will write $\bigcap A_2 \cap A_4 \cap A_5$ as $A_2 A_4 A_5$.

Note: Recall that a sequence is just a function with domain $\{1, 2, 3, \ldots, n\}$. The subsequences of $f$ are obtained by restricting $f$ to the different subsets of $\{1, 2, 3, \ldots, n\}$. So from the sequence $\langle f(1), f(2), \ldots, f(n) \rangle$ we can get $\langle \rangle$ by restricting $f$ to $\emptyset$, $\langle f(2), f(3) \rangle$ by restricting $f$ to $\{2, 3\}$, $\langle f(1), f(3), f(4) \rangle$ by restricting $f$ to $\{1, 3, 4\}$. Since there are $2^n$ subsets of $\{1, 2, 3, \ldots, n\}$, there will be $2^n$ different subsequences of $\langle f(1), f(2), \ldots, f(n) \rangle$. This immediately tells us that there will be $2^n$ positive sets because there are $2^n$ subsequences $\langle i_1, \ldots, i_k \rangle$ of $\langle 1, 2, 3, \ldots, n \rangle$.

Let us analyze the positive sets in more details.
Def. The order of a positive set is the number of $A_i$'s it contains. In other words, it is the length of the subsequence $(i_1, \ldots, i_k)$ from which it came.

Prop. 1. There are $\binom{n}{k}$ positive sets of order $k$.

(a) Consequently, there are (again) $2^n$ positive sets.

Proof. (a) The number of positive sets of order $k$ is the number of subsequences $(i_1, \ldots, i_k)$ of $(1, 2, 3, \ldots, n)$ with $k$ terms. But this is just the number of $k$-subsets of $\{1, 2, \ldots, n\}$ because $(i_1, \ldots, i_k)$ has to be in increasing order. Since there are $\binom{n}{k}$ $k$-subsets of $\{1, 2, \ldots, n\}$, there will be $\binom{n}{k}$ positive sets of order $k$.

(b) The number of positive sets is equal to

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n$$

by a previous result.

Positive sets of Order 0: $U$.

Order 1: $A_1, A_2, \ldots, A_n$.

Order 2: $A_1A_2, A_1A_3, \ldots, A_{n-1}A_n$.

Order n: $A_1A_2A_3\cdots A_n$.

Order 2 (in more details): $A_1A_2, A_1A_3, A_1A_4, \cdots, A_1A_n$.

Order 3: $A_2A_3, A_2A_4, \cdots, A_2A_n, A_3A_4, A_3A_5, \cdots, A_3A_n$.

Order 4: $A_4A_5, A_4A_6, \cdots, A_4A_n, \cdots, \frac{A_{n-2}A_{n-1}}{2}, \frac{A_{n-2}A_n}{2}, \frac{A_{n-1}A_n}{1}$.
So the number of positive sets of order 2 will be \((n-1)+(n-2)+(n-3)+\ldots+2+1 = \frac{n(n-1)}{2}\) as indicated before.

Theorem 2 (Inclusion-Exclusion Theorem - First version)

Let \(U\) be a universal set and \(A_i = \{x \in U: x \text{ has property } P_i\}\) for \(i=1,2,\ldots,n\). Then the number of elements of \(U\) with none of the properties \(P_i\) is given by

\[
\left|A_1^cA_2^cA_3^c\cdots A_n^c\right| = \sum_{k=0}^{n} (-1)^k \left\{ \sum_{1 \leq i_1 < \ldots < i_k \leq n} |A_{i_1}A_{i_2}\cdots A_{i_k}| \right\}
\]

\[
= |U| - \sum_{1 \leq i \leq n} |A_i| + \sum_{1 \leq i < j \leq n} |A_iA_j| - \ldots \]

\[
+ (-1)^k \sum_{1 \leq i_1 < \ldots < i_k \leq n} |A_{i_1}\cdots A_{i_k}| + \ldots + (-1)^n |A_1A_2\cdots A_n| .
\]

Proof: We shall prove \((*)\) by showing that an element of \(U\) with none of the properties \(P_1, \ldots, P_n\) is counted once in the RHS \((*)\) and that an element with at least one of the properties \(P_1, \ldots, P_n\) is counted zero times in the RHS \((*)\).

Now if \(x\) has none of the properties \(P_1, \ldots, P_n\) then \(x\) will be counted exactly once in \(|U|\) and zero times in all the other terms of the RHS \((*)\).

Also if \(x\) has the properties \(P_{i_1}, P_{i_2}, \ldots, P_{i_k}\)
then $x$ will be counted in the RHS($x$) $\binom{k}{0} - \binom{k}{1} + \binom{k}{2} - \binom{k}{3} + \cdots + (-1)^k \binom{k}{k} = 0$ times.

No. of times $x$ is counted
in the sets of order 1

No. of times $x$ is counted
in the sets of order 2

No. of times $x$ is counted
in the sets of order $k$.

\[ \therefore \text{LHS}(\ast) = \text{RHS}(\ast), \text{ So the result follows.} \]

Corollary 3 (Inclusion-Exclusion Theorem - Version 2)

Let $A_1, A_2, \ldots, A_n$ be as in Theorem 2. Then

\[ |A_1 \cup A_2 \cup \cdots \cup A_n| = \sum_{k=1}^{n} (-1)^{k-1} \sum_{1 \leq i_1 < \cdots < i_k \leq n} |A_{i_1} \cap A_{i_2} \cdots \cap A_{i_k}| \]

\[ = \sum_{1 \leq i \leq n} |A_i| - \sum_{1 \leq i < j \leq n} |A_i \cap A_j| + \cdots + (-1)^{k-1} \sum_{1 \leq i_1 < \cdots < i_k \leq n} |A_{i_1} \cdots \cap A_{i_k}| \]

\[ + \cdots + (-1)^{n-1} |A_1 \cap A_2 \cap \cdots \cap A_n| \]

Proof: We know that

\[ |A_1 \cup A_2 \cup \cdots \cup A_n| = |U| - |(A_1 \cup A_2 \cup \cdots \cup A_n)^c| \]

\[ = |U| - |A_1^c \cap A_2^c \cap \cdots \cap A_n^c| \]

\[ = |U| - |A_1^c \cap A_2^c \cap A_3^c \cdots \cap A_n^c| \]

\[ = |U| - \text{RHS(\ast)} \text{ of Theorem 2} \]

\[ = |U| - \sum_{k=0}^{n} (-1)^k \sum_{1 \leq i_1 < \cdots < i_k \leq n} |A_{i_1} \cap A_{i_2} \cdots \cap A_{i_k}| \]

\[ = \sum_{1 \leq i \leq n} |A_i| - \sum_{1 \leq i < j \leq n} |A_i \cap A_j| + \cdots + (-1)^{k-1} \sum_{1 \leq i_1 < \cdots < i_k \leq n} |A_{i_1} \cdots \cap A_{i_k}| \]

\[ + \cdots + (-1)^{n-1} |A_1 \cap A_2 \cap \cdots \cap A_n| \].
§2. Two forbidden position problems

Recall that a permutation of \( \{1, 2, 3, \ldots, n\} \) is a sequence of \( n \) distinct elements of \( \{1, 2, 3, \ldots, n\} \). In other words, a permutation of \( \{1, 2, 3, \ldots, n\} \) is a bijection (one-to-one correspondence) from \( \{1, 2, 3, \ldots, n\} \) to itself.

So the permutation \( (2, 3, 1) \) is really the bijection \( (1 \ 2 \ 3) \), i.e., the function which sends 1 to 2, 2 to 3, and 3 to 1.

**Def.** A derangement of \( \{1, 2, \ldots, n\} \) is a permutation of \( \{1, 2, \ldots, n\} \) in which no element is in its natural position, i.e., in which no element goes to itself.

**Ex.1** \( (2, 3, 1) \) & \( (3, 1, 2) \) are the derangements of \( \{1, 2, 3\} \), \( (2, 1, 3), (1, 3, 2), (3, 2, 1) \) & \( (1, 2, 3) \) are not derangements of \( \{1, 2, 3\} \).

**Ex.2** Let \( D_n \) = set of all derangements of \( \{1, 2, \ldots, n\} \) and \( D_0 = \{\} \). Then
\[
\begin{align*}
D_0 &= \{\} \quad \text{so} \quad D_0 = 1 \\
D_1 &= \emptyset \quad D_1 = 0 \\
D_2 &= \{(2, 1)\} \quad D_2 = 1 \\
D_3 &= \{(2, 3, 1), (3, 1, 2)\} \quad \text{and} \quad D_3 = 2 \\
\end{align*}
\]
We will later see that \( D_4 = 9 \).
Theorem 4: The number of derangements of \(\{1, 2, \ldots, n\}\) is given by

\[
D_n = n! \left\{ \frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!} \right\}
\]

\[
= n! \left[ \sum_{k=0}^{n} (-1)^k / k! \right]
\]

Proof:

Let \(U\) = set of all permutations of \(\{1, 2, \ldots, n\}\)

Put \(A_i\) = set of all permutations in \(U\) with \(i\) going to itself, \(i = 1, \ldots, n\).

Then \(|U| = n!

\[|A_i| = (n-1)! \quad \left( \begin{array}{c} 1 \ 2 \ \cdots \ i \ \cdots \ n \\ \end{array} \right) \]

Also \(A_i A_j = A_i \cap A_j\)

= set of all permutations in \(U\) with \(i\) going to \(i\) & \(j\) going to \(j\). So \(\quad |A_i A_j| = (n-2)! \quad \text{for} \ 1 \leq i < j \leq n.\)

In general \(|A_i A_i^2 \cdots A_i^k| = (n-k)! \quad (\text{for} \ 1 \leq i_1 < i_2 \leq \cdots < i_k \leq n)\) for each of the \(\binom{n}{k}\) positive sets of order \(k\): So by the Inclusion-Exclusion Theorem we get

\[
D_n = |A_1^c \cap A_2^c \cap \cdots \cap A_n^c| = |A_1^c A_2^c \cdots A_n^c|
\]

\[
= \sum_{k=0}^{n} (-1)^k \left\{ \sum_{1 \leq i_1 < \cdots < i_k \leq n} |A_{i_1} A_{i_2} \cdots A_{i_k}| \right\}
\]

\[
= \sum_{k=0}^{n} (-1)^k \left\{ \binom{n}{k} \cdot (n-k)! \right\} = n! \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}
\]

\[
= n! \left\{ \frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!} \right\}.
\]
Prop 5. (a) For any $n \geq 1$, \[ D_n = \binom{n}{m} D_{n-1} + (-1)^m \]
(b) For any $n \geq 2$, \[ D_n = (n-1)(D_{n-1} + D_{n-2}) \]

Proof: (a) \[ D_n = n! \left[ \frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots + (-1)^{n-1} + \frac{(-1)^n}{n!} \right] = n! \left[ \frac{1}{0!} + \frac{1}{1!} - \frac{1}{2!} - \frac{1}{3!} + \ldots + \frac{(-1)^{n-1}}{(n-1)!} + (-1)^n \right] = n! \left( D_{n-1} + (-1)^n \right) \]

(b) \[ D_n = \left[n, D_{n-1}\right] + (-1)^n = (n-1) D_{n-1} + D_{n-1} + (-1)^n = (n-1) D_{n-1} + \left[(n-1) D_{n-2} + (-1)^{n-1}\right] + (-1)^n = (n-1) D_{n-1} + (n-1) D_{n-2} + (-1)^{n-1} \left[1 - 1\right] = (n-1) \left[ D_{n-1} + D_{n-2} \right] \]

Ex 3. We have already seen that $D_3 = 2$. So
\[ D_4 = 4, D_3 + (-1)^4 = 4(2) + 1 = 9 \]
\[ D_5 = 5, D_4 + (-1)^5 = 5(9)(-1) = 44 \]
\[ D_6 = 6, D_5 + (-1)^6 = 6(44) + 1 = 265 \]
\[ D_7 = 7, D_6 + (-1)^7 = 7(265)(-1) = 1854 \]

Ex 4. In how many ways can we return the watches of 3 men and 3 ladies so that
(a) no person gets their own watch
(b) no person gets their own watch and each lady receives a ladies watch.

Sol. (a) Answer = $D_6 = 265$ ways
(b) Answer = $(D_3)(D_3) = 2(2) = 4$ ways, bec.
each lady will get a ladies watch & the men will get men's watches.
Ex. 5. In how many ways can we return the
cars of 5 super-models so that
(a) no supermodel gets her own car
(b) exactly one supermodel gets her own car
(c) exactly two super-models get her own car
(d) at most two super-models get her own car
(e) at least two super-models get her own car

(a) \text{Answer} = \lambda_5 = 44

(b) There are \(\binom{5}{1}\) ways to choose the one super-
model who will get her own car. Then we
derange the cars of the other 4 super-models
in \(\lambda_4\) ways. So our answer = \(\binom{5}{1} \cdot \lambda_4 = 45\).

(c) There \(\binom{5}{2}\) ways to choose the two super-models
who will get their own cars. Then we derange
the cars of the other 3 super-models in \(\lambda_3\) ways
Answer will be \(\binom{5}{2} \cdot \lambda_3 = \frac{5!4}{2!} \cdot 2 = 20\).

(d) \text{Answer} = \text{Ans(a)} + \text{Ans(b)} + \text{Ans(c)}

\[= 44 + 45 + 20 = 109\]

(e) \text{Answer} = \text{total no. of ways of permuting the cars}

\[= 5! - \text{[Ans(a) + Ans(b)]}\]
\[= 120 - (44 + 45) = 31\]

(e') We can also add the number of ways 2, 3, 4
& 5 super-models get their own cars to get
\[\text{Ans(e)} = \binom{5}{2} \cdot \lambda_3 + \binom{5}{3} \cdot \lambda_2 + \binom{5}{4} \cdot \lambda_1 + \binom{5}{5} \cdot \lambda_0\]
\[= 10(2) + 10(1) + 5(0) + 1(1)\]
\[= 20 + 10 + 1 = 31\]
Def. A non-consecutive permutation of \( \{1, 2, \ldots, n\} \) is a permutation of \( \{1, 2, \ldots, n\} \) in which there is no pair of consecutive terms of the form \( \langle i, i+1 \rangle \). In other words, if we view the permutation as a bijection, then there is no value of \( j \) such that \( f(j+1) = f(j) + 1 \), for \( j = 1, 2, \ldots, n-1 \).

Ex. 6 (a) \( \langle 1, 3, 2 \rangle \), \( \langle 2, 1, 3 \rangle \), and \( \langle 3, 2, 1 \rangle \) are non-consecutive permutations of \( \{1, 2, 3\} \). (b) \( \langle 2, 1, 3 \rangle \), \( \langle 2, 3, 1 \rangle \), and \( \langle 3, 1, 2 \rangle \) are not non-consecutive permutations of \( \{1, 2, 3\} \).

Notation: Let \( Q_n \) = set of all non-consecutive permutations of \( \{1, 2, \ldots, n\} \) and \( Q_n = |Q_n| \). Then

\[
Q_0 = \{\langle \rangle \} \quad \Rightarrow \quad Q_0 = 1
\]

\[
Q_1 = \{\langle 1 \rangle \} \quad \Rightarrow \quad Q_1 = 1
\]

\[
Q_2 = \{\langle 2, 1 \rangle \} \quad \Rightarrow \quad Q_2 = 1
\]

\[
Q_3 = \{\langle 1, 3, 2 \rangle, \langle 2, 1, 3 \rangle, \langle 3, 2, 1 \rangle \} \quad \text{and} \quad Q_3 = 3
\]

Later on we will see that \( Q_4 = 12 \).

Theorem: The number of non-consecutive permutations of \( \{1, \ldots, n\} \) is

\[
Q_n = \binom{n}{n-1} n! - \binom{n-1}{n-2} (n-1)! + \binom{n-2}{n-3} (n-2)! - \cdots + (-1)^{n-1} \binom{1}{n-1} 1!
\]

\[
= \sum_{k=0}^{n-1} \binom{n-1}{k} (-1)^k (n-k)!
\]

Proof: Let \( U = \) set of all permutations of \( \{1, 2, \ldots, n\} \) and \( A_i = \) set of all permutations in \( U \) which contain \( \langle i, i+1 \rangle \) as consecutive terms.
Then $A_1 = \text{set of permutations in } U \text{ with } \langle 1, 2 \rangle$
as a pair of consecutive terms
$$= \text{set of permutations of } \{1, 2, 3, 4, \ldots, n\}$$
So $|A_1| = (n-1)!$. Similarly, $|A_2| = (n-1)!$
for $i = 2, \ldots, n-1$ as well.

Also $A_1A_2 = \text{set of permutations in } U \text{ with both}$
$\langle 1, 2 \rangle$ and $\langle 2, 3 \rangle$ as pairs of consecutive terms
$$= \text{set of permutations of } \{1, 2, 3, 4, \ldots, n\}$$
So $|A_1A_2| = (n-2)!$

And $A_1A_3 = \text{set of permutations in } U \text{ with both}$
$\langle 1, 2 \rangle$ and $\langle 3, 4 \rangle$ as pairs of consecutive terms
$$= \text{set of permutations of } \{1, 2, 3, 4, \ldots, n\}$$
So $|A_1A_3| = (n-2)!$

From this we can see that for any $i$ and $j$ with
$1 \leq i < j \leq n-1$, we have $|A_i, A_j| = (n-2)!$

In general we can also see that for any
$\langle i_1, \ldots, i_k \rangle$ with $1 \leq i_1 < i_2 < \ldots < i_k \leq n-1$, we have
$|A_{i_1}A_{i_2}\ldots A_{i_k}| = (n-k)!$
So $Q_n = \text{set of all permutations in } U \text{ with}$
no pair of consecutive terms
$$= |A_1^cA_2^c\ldots A_{n-1}^c| = \sum_{k=0}^{n-1} (-1)^k \left\{ \sum_{1 \leq i_1 < \ldots < i_k \leq n-1} |A_{i_1}\ldots A_{i_k}| \right\}$$
$$= \sum_{k=0}^{n-1} (-1)^k \left( \binom{n-1}{k} \cdot (n-k)! \right)$$
$$= (n-1)! - \binom{n-1}{1} \cdot (n-1)! + \binom{n-1}{2} \cdot (n-2)! - \ldots + (-1)^{n-1} \cdot (n-1)!!.$$
Prop. 7 For any \( n \geq 1 \), \( Q_n = \Delta n + \Delta n_{n-1} \).

Proof: \( Q_n = \sum_{k=0}^{n-1} (-1)^k \frac{(n-1)!}{k!(n-1-k)!} \frac{(n-k)!}{k!} \)

\[ = \sum_{k=0}^{n-1} (-1)^k \frac{(n-1)!}{k!(n-1-k)!} \left( n - \sum_{k=0}^{n-1} (-1)^k \frac{(n-1)!}{k!(n-1-k)!} \right) \]

\[ = \sum_{k=0}^{n-1} (-1)^k \frac{n!}{k!} - \sum_{k=0}^{n-1} (-1)^k \frac{(n-1)!}{k!(n-1-k)!} \frac{n!}{k!} \]

\[ = \frac{\sum_{k=0}^{n-1} (-1)^k \frac{n!}{k!}}{k!} - (-1)^n \frac{n!}{k!} - \sum_{k=0}^{n-2} (-1)^k \frac{(n-1)!}{k!(n-1-k)!} \frac{n!}{k!} \]

\[ = n! \left[ \sum_{k=0}^{n-1} \frac{(-1)^k}{k!} \right] - (-1)^n \frac{n!}{k!} + \sum_{k=0}^{n-1} (-1)^k \frac{(n-1)!}{k!(n-1-k)!} \frac{n!}{k!} \]

\[ = n! \left[ \sum_{k=0}^{n-1} \frac{(-1)^k}{k!} \right] + (n-1)! \left[ \sum_{k=0}^{n-1} \frac{(-1)^k}{k!} \right] - (-1)^n n! \frac{(-1+1)}{2} \]

\[ = \Delta n + \Delta n_{n-1}. \]

Ex. 6 Five sisters walk to school in a straight line. In how many ways can they walk back home in a straight line so that no sister sees the same person in front of them again.

Sol. Answer = \( Q_5 = \Delta 5 + \Delta 4 = 44 + 9 = 53 \) ways.
Ex. 1 How many integer-solutions of the equation 
\[ x_1 + x_2 + x_3 = 17 \]
are there with \( x_1 \geq 3 \), \( x_2 \geq 5 \), and \( x_3 \geq 2 \)?

Sol. Let \( x_1 = y_1 + 3 \), \( x_2 = y_2 + 5 \), and \( x_3 = y_3 + 2 \).
Then our answer will be the same as the number of integer-solutions of the equation
\[ (y_1 + 3) + (y_2 + 5) + (y_3 + 2) = 17 \]
with \( y_1 + 3 \geq 3 \), \( y_2 + 5 \geq 5 \), and with \( y_3 + 2 \geq 2 \).

This is the same as the number of integer-solutions of \( y_1 + y_2 + y_3 = 7 \) with \( y_1 \geq 0 \), \( y_2 \geq 0 \), \( y_3 \geq 0 \).
And we know that this is the same as the number of ways of arranging 8 1's & 2 5's
in a row, i.e., \( \frac{(7+2)!}{7!2!} = \binom{9}{3} - 1 = \binom{9}{2} \)
So our final answer is \( \binom{9}{2} = \frac{9!8}{2!7!} = 36 \).

Ex. 2 Let \( M = \{ a, a, a, b, b, b, c, c, c \} \). How many 17-combinations of \( M \) are there with \( \geq 3 \) a's,
\( \geq 5 \) b's, and \( \geq 2 \) c's?

Sol. Let \( x_1 = \text{no. of a's in a 17-comb. of } M \)
\( x_2 = \text{no. of b's in the same 17-comb. of } M \)
and \( x_3 = \text{no. of c's in the same 17-comb. of } M \).
Then our answer to the problem will be the number of integer-solutions of the equation
\( x_1 + x_2 + x_3 = 17 \) with \( x_1 \geq 3 \), \( x_2 \geq 5 \), and \( x_3 \geq 2 \).

And from Example 1, we found that this is 36.

**Sol. 2**

Now there is another way to do this problem.

Let \( A = \) set of all 7-comb. of \( M \) and

\[ A' = \text{set of each 7-comb. in } A \text{ plus } [3a, 5b, 2c] \]

Then \( A' \) is a 17-comb. of \( M \) because each element of \( A' \) was obtained by adding a multi-set with 10 elements to a 7-comb. of \( M \). Also each element of \( A' \) is a 17-comb.

of \( M \) with \( \geq 3 \) a’s, \( \geq 5 \) b’s, and \( \geq 2 \) c’s.

Since there is an obvious bijection from \( A \) to \( A' \), it follows that

\[ |A'| = \text{No. of 17-comb. of } M \text{ with } \geq 3 \text{ a’s}, \geq 5 \text{ b’s, } \geq 2 \text{ c’s} \]

\[ = \text{No. of 7-comb. of } M = |A| = (7+3-1) \]

So, our final answer is \( (9) = \frac{9!}{3!} = 36 \) again.

**Ex. 3**

How many 15-combinations of the finite multi-set \( F = [4a, 6b, 20c] \) are there?

**Sol.**

Let \( M = [\infty \text{a, } \infty \text{b, } \infty \text{c}] \) and put

\( U = \) set of all 15-combinations of \( M \)

\( A = \) set of all 15-comb. in \( U \) with > 4 a’s,

\( B = \) set of all 15-comb. in \( U \) with > 6 b’s,

\( \& C = \) set of all 15-comb. in \( U \) with > 20 c’s.

Then

\( A = \) set of all 10-comb. of \( M \) with 5 extra a’s added

\( B = \) set of all 8-comb. of \( M \) with 7 extra b’s added

\( \& C = \emptyset \), bec. a 15-comb. cannot have >21 c’s.
So \[|U| = \binom{15+3-1}{3-1}, \quad |A| = \binom{10+3-1}{3-1}, \quad B = \binom{8+3-1}{3-1}\] and \[k = 0.

Also \(A \cap B\) = set of all 15-comb. of \(M\) with \(\geq 5a's\) & \(\geq 7b's\),

\(=\) set of all 3-comb. of \(M\) with \([5a, 7b]\) added.

\([A \cap B] = \binom{3+3-1}{3-1}.\) Since we want \(\leq 4a's\), \(\leq 6b's\) and \(\leq 20c's\) in our 15-comb. of \(M\),

our final answer would be \([A \cap B] \cap C^{C}\).

But by the Inclusion-Exclusion Theorem

\[|A \cap B \cap C^{C}| = |U| - |A| - |B| - |C| + |A \cap B| + |A \cap C|
+ |B \cap C| - |A \cap B \cap C|\]

\[= \binom{17}{2} - \binom{12}{2} - \binom{10}{2} + \binom{5}{2} + 0 + 0 - 0\]

\[= \binom{17}{2} + \binom{5}{2} - \binom{12}{2} - \binom{10}{2}\]

because \(C, A \cap C, B \cap C\)

and \(A \cap B \cap C\) are all empty.

\[= \frac{17(16)}{2} + \frac{5(4)}{2} - \frac{12(11)}{2} - \frac{10(9)}{2} = 136 + 10 - 66 + 45 = 35.\]

**Ex. 4** How many 26-comb. of the finite multi-set \(F = [4.a, 6.b, 20.c]\) are there?

**Sol. 1** Again let \(M = [\infty.a, \infty.b, \infty.c]\) and put

\(U =\) set of all 26-comb. of \(M\)

\(A =\) set of all 26-comb. in \(U\) with \(\geq 4a's\) \(\geq 5a's\)

\(B =\) set of all 26-comb. in \(U\) with \(\geq 6b's\)

\(C =\) set of all 26-comb. in \(U\) with \(\geq 20c's\)

Then

\(A =\) set of all 21-comb. of \(M\) with \([5a]\) added to each 21-comb.

\(B =\) set of all 19-comb. of \(M\) with \([7b]\) added to each 19-comb.

\(C =\) set of all 5-comb. of \(M\) with \([21c]\) added to each 5-comb.

\(A \cap B =\) set of all 14-comb. of \(M\) with \([5a, 7b]\) added to each 14-comb.
\( A_{nc} = \) set of all 0-comb. of \( M \) with \([5a, 21c]\) added to each 0-comb.

\( B_{nc} = \emptyset \) and \( A \cap B_{nc} = \emptyset \). So

\[
\text{Number of 26-comb. of } F = |A^c \cap B^c \cap C^c| = (26 + 3 - 1) - (21 + 3 - 1) - (19 + 3 - 1) + (14 + 3 - 1) + (8 + 3 - 1) + 0 - 0
\]

\[
= \binom{28}{2} + \binom{16}{2} + \binom{2}{2} - \binom{23}{2} - \binom{21}{2} - \binom{9}{2} = 499 - 484 = 15.
\]

**Sol. 2** But there is a much quicker way to do the same problem. We just have to observe that No. of 26-comb. of \( F \) = No. of 4-comb. of \( F \) because \( F \) has 30 elements. If we want to pick 26 elements out of \( F \), we can just pick 4 Elements to leave behind and get the same answer. So let \( M = \{00a, 00b, 00c\} \) & \( U = \) set of all 4-comb. of \( M \). Put \( A = \) set of all 4-comb. in \( U \) with \( > 4 \) a's, \( B = \) set of all 4-comb. in \( U \) with \( > 6 \) b's, & \( C = \) set of all 4-comb. in \( U \) with \( > 20 \) c's. Then \( A = B = C = \emptyset \) and \( A \cap B = A \cap C = B \cap C = A \cap B \cap C = \emptyset \) also. So

\[
\text{Number of 4-comb. of } F = |A^c \cap B^c \cap C^c| = (4 + 3 - 1) - (3 - 1) = \binom{6}{2} = 15 \text{ (as before)}
\]

So as you can see, it pays to be a little smart and think a little bit before trying to solve the problem. By the way, this trick would not have worked with Ex. 3.
Def. Let $U$ be a universal set and $A_1, \ldots, A_n$ be subsets of $U$. An ultimate set with respect to $A_1, \ldots, A_n$ is any set of the form $X_1 \cap X_2 \cap \ldots \cap X_n$ where $X_i = A_i$ or $A_i^c$ for $i = 1, \ldots, n$.

Prop. 8 There are $2^n$ ultimate sets w.r.t. $A_1, \ldots, A_n$.

Proof. For each $X_i$ we have 2 choices. Since there are $n$ $X_i$'s we will get $2^n$ choices & so $2^n$ ultimate sets.

Ex. 5 Let $U$ be a universal and $A_1, A_2$ be subsets of $U$. Find all the ultimate sets w.r.t. $A_1, A_2$.

Sol. They are $A_1 \cap A_2$, $A_1 \cap A_2^c$, $A_1^c \cap A_2$, $A_1^c \cap A_2^c$.

Prop. 9 The ultimate sets w.r.t. $A_1, \ldots, A_n$ are all pairwise disjoint.

Proof. Suppose $Z_1$ and $Z_2$ are ultimate sets. Let $Z_1 = X_1 \cap X_2 \cap \ldots \cap X_n$ and $Z_2 = Y_1 \cap Y_2 \cap \ldots \cap Y_n$ where $X_i = A_i$ or $A_i^c$, and $Y_i = A_i$ or $A_i^c$.

Then for some $i_o$, $X_{i_o}$ & $Y_{i_o}$ must be different (because if $X_i = Y_i$ for each $i$, then $Z_1$ & $Z_2$ would be the same). Hence $Z_1 \cap Z_2 = (X_1 \cap X_2 \cap \ldots \cap X_n) \cap (Y_1 \cap Y_2 \cap \ldots \cap Y_n) \subseteq X_{i_o} \cap Y_{i_o} = \emptyset$.

i.e. $Z_1 \cap Z_2 = \emptyset$. Hence any two ultimate sets are disjoint.
Consistency of Data

EX. 6  Suppose we are told: among the Math majors
(a) 26 of them are taking Graph Theory
    or Combinatorics, or both;
(b) 12 of them are taking Graph Theory
(c) 18 of them are taking Combinatorics
(d) 15 of them are not taking Combinatorics

Determine whether or not this data is consistent.

Let $U =$ set of Math majors, $A =$ set of Math majors taking Graph Theory, and $B =$ set of math majors taking Combinatorics. Put

- $x_1 = |A\cap B|$
- $x_2 = |A \cap B^c|$
- $x_3 = |A^c \cap B|$
- $x_4 = |A^c \cap B^c|$

Then the data translates to the system of equations

\[
\begin{align*}
 x_1 + x_2 + x_3 &= 20 \quad (1) \\
 x_1 + x_2 &= 12 \quad (2) \\
 x_1 + x_3 &= 18 \quad (3) \\
 x_2 + x_4 &= 15 \quad (4)
\end{align*}
\]

Then there will be 3 possibilities.

I: The system has no solution. In this case, the data will be inconsistent.

IIA: The system has a unique solution. In this case the data is consistent & it determines the situation.

IIB: The system has more than one solutions. In this case the data is consistent but it does not determine the situation.

In Ex 6 the system has a solution. So the data is indeed consistent.

- $0 - (2) \Rightarrow x_3 = 8,$
- $(3) + (4) - (1) \Rightarrow x_4 = 13,$
- $(3) \Rightarrow x_1 = 18 - x_3 = 10.$
- $(2) \Rightarrow x_2 = 12 - x_1 = 2.$

\[ x_1 = 10, x_2 = 2, x_3 = 8, x_4 = 13 \]

\[ \text{END OF CH. 4.} \]