§ 1. The Difference Operator

Recall that if \( \{x_n\}_{n \in \mathbb{N}} \) was a sequence, we defined the operator \( E \) by \( E(\{x_n\}) = \{x_{n+1}\}_{n \in \mathbb{N}} \).

Def. We define the difference operator \( \Delta \) by \( \Delta = E - I \).

So \( \Delta (\{x_n\}) = (E - I)(\{x_n\}) = \{x_{n+1} - x_n\}_{n \in \mathbb{N}} \).

Ex. 1. Let \( \{x_n\} = \{n^2 + n + 1\}_{n \in \mathbb{N}} \). Then
\[
\langle x_n \rangle = 1, 5, 11, 19, 29, 41, \ldots \\
\langle \Delta x_n \rangle = 4, 6, 8, 10, 12, 14, \ldots \\
= 4, 6, 8, 10, 12, 14, \ldots 
\]
We can also write \( \Delta x_n \) just as \( \langle x_{n+1} - x_n \rangle \).

So \( \Delta x_n = x_{n+1} - x_n = \left\lfloor \frac{(n+1)^2 + 3(n+1)+1}{2(n+1)} \right\rfloor - \left\lfloor \frac{n^2 + 3n + 1}{2n} \right\rfloor = \frac{(n^2 + 2n + 1 + 3n + 2) - (n^2 + 3n)}{2n+4} = \frac{2n+4}{2n+4} = 1 \).

Def. We define \( \Delta^k \) by recursion as follows.

\[
\Delta^0 = I, \\
\Delta^{k+1} = \Delta(\Delta^k), \quad \text{for } k \geq 0. 
\]
In particular, \( \Delta^2 = \Delta(\Delta) = (E - I)(E - I) = E^2 - 2E + I \).

\( \Delta^3 = (E-I)^3 = E^3 - 3E^2 + 3E - I \).

\[
\langle \Delta^2 (x_n) \rangle = \langle x_{n+2} - 2x_{n+1} + x_n \rangle \\
\langle \Delta^3 (x_n) \rangle = \langle x_{n+3} - 3x_{n+2} + 3x_{n+1} - x_n \rangle. 
\]

Ex. 2. Let \( \{x_n\} = \{n^2 + 3n + 1\}_{n \in \mathbb{N}} \). Find \( \langle \Delta x_n \rangle, \langle \Delta^2 x_n \rangle \) and \( \langle \Delta^3 x_n \rangle \).
\[ \langle n \rangle = 0, 1, 2, 3, 4, 5, 6, \ldots \]

\[ \langle \Delta n \rangle = 1, 5, 11, 19, 29, 41, 55, \ldots \]

\[ \langle \Delta^2 n \rangle = 2, 2, 2, 2, 2, 2, 2, \ldots \]

\[ \langle \Delta^3 n \rangle = 0, 0, 0, 0, 0, 0, 0, \ldots \]

Prop. 1 Let \( x_n = P_k(n) \), where \( P_k(n) \) is a polynomial of degree \( k \). Then \( \langle \Delta^{k+1} x_n \rangle = \langle 0 \rangle_{n \in \mathbb{N}}. \)

Proof: We will prove the result by induction on \( k \).

(a) If \( k = 0 \), then \( P_k(n) = a_0 \), where \( a_0 \) is a constant. So \( \Delta x_n = x_{n+1} - x_n = a_0 - a_0 = 0 \). Hence the result is true when \( k = 0 \).

(b) Suppose the result is true for \( k \) where \( k > 0 \).

Then \( \langle \Delta^k P_k(n) \rangle = \langle 0 \rangle_{n \in \mathbb{N}} \) for any polynomial \( P_k(n) \) of degree \( k \). Now let \( P_{k+1}(n) \) be any polynomial in \( n \) of degree \( k+1 \). Then

\[ P_{k+1}(n) = a_{k+1} n^{k+1} + a_k n^k + \cdots + a_1 n + a_0. \]

So

\[ \Delta P_{k+1}(n) = a_{k+1} \left[ (n+1)^{k+1} - n^{k+1} \right] + a_k \left[ (n+1)^k - n^k \right] + \cdots + a_1 [n(n+1) - n] + [a_0 - a_0] \]

\[ = a_{k+1} \left[ (k+1)n^{k+1} + \cdots + (k+1)n + 1 - n^{k+1} \right] \]

\[ + a_k \left[ (n+1)^k - n^k \right] + \cdots + a_1. \]

Thus

\[ \Delta^{k+1}(P_{k+1}(n)) = \Delta^k (\Delta P_{k+1}(n)) = \Delta^k (\langle 0 \rangle) = \langle 0 \rangle \]

because of the induction hypothesis. So if the result is true for \( k \), it will be true for \( k+1 \).

(c) By the Principle of Math Induction, it follows that the result is true for all \( k \).
**Def.** The zero-column of an infinite sequence \( \langle x_n \rangle_{n \in \mathbb{N}} \) is the sequence \( \langle \Delta^k x_0 \rangle_{k \in \mathbb{N}} \). (Some authors say zero-diagonal instead of zero-column).

**Ex. 3** Let \( \langle x_n \rangle_{n \in \mathbb{N}} = \langle n^2 + 3n + 1 \rangle_{n \in \mathbb{N}} \). Then the zero-column of \( \langle x_n \rangle \) is \( \langle 1, 4, 7, 10, ... \rangle \) as shown in Ex. 2. The interesting thing is that
\[
\begin{align*}
1 \cdot \binom{n}{0} + 4 \cdot \binom{n}{1} + 2 \cdot \binom{n}{2} &= 1 + 4n + 2n(n-1) \\
&= 1 + 4n + (n^2 - n) \\
&= 1 + 3n + n^2 = x_n.
\end{align*}
\]
This fact is true whenever \( x_n \) is a polynomial in \( n \).

**Prop.** Let \( \langle x_n \rangle_{n \in \mathbb{N}} \) be an infinite sequence and suppose that \( \langle \Delta^k x_0 \rangle_{k \in \mathbb{N}} = \langle c_0, c_1, ..., c_p, 0, 0, ... \rangle \) where all the terms after \( c_p \) are all zeros. Then
\[ x_n = c_0 \binom{n}{0} + c_1 \binom{n}{1} + c_2 \binom{n}{2} + ... + c_p \binom{n}{p} = \sum_{k=0}^{p} \frac{k^k x_0}{k!} b_k. \]

**Proof.** See textbook. Here \( [n]_k = n(n-1)(n-2)...(n-k+1) \).

It is nice to know that if \( x_n \) is a polynomial of degree \( p \), then \( x_n \) can be expressed as
\[ x_n = c_0 \binom{n}{0} + c_1 \binom{n}{1} + c_2 \binom{n}{2} + ... + c_p \binom{n}{p}, \]
but the main reason for us to express \( x_n \) in this form is to be able to find \( \sum_{k=1}^{n} x_k \).

**Ex. 4** Let \( x_n = 1 + 3n + n^2 \). Then from Example 3
\[
\begin{align*}
x_n &= 1 \cdot \binom{n}{0} + 4 \cdot \binom{n}{1} + 2 \cdot \binom{n}{2} \\
\sum_{k=0}^{n} x_k &= \sum_{k=1}^{n} \left[ 1 \binom{k}{0} + 4 \binom{k}{1} + 2 \binom{k}{2} \right]
\end{align*}
\]
Ex. 4

\[ E = 1 + \sum_{k=0}^{n} \binom{n}{k} + 4 \sum_{k=0}^{n} \binom{n+1}{k} + 2 \sum_{k=0}^{n} \binom{n+1}{k} \]

\[ = 1 \cdot \binom{n+1}{1} + 4 \cdot \binom{n+1}{2} + 2 \cdot \binom{n+1}{3} \]

\[ = (n+1) + 4 \cdot \frac{(n+1)\cdot(n)}{2} + 2 \cdot \frac{(n+1)\cdot(n)\cdot(n-1)}{3} \]

\[ = \frac{(n+1)}{3} \left[ 1 + 2n + n(n-1) \right] = \frac{(n+1)(n^2 + 5n + 3)}{3} \]

Let us check:

\[ 1 + \frac{(n+1)(n^2 + 5n + 3)}{3} = 1 \]

\[ 1 + \frac{(n+1)(n^2 + 5n + 3)}{3} = 2(n+1) = 6 \]

\[ 1 + \frac{(n+1)(n^2 + 5n + 3)}{3} = 3(n+1) = 17 \]

\[ 1 + \frac{(n+1)(n^2 + 5n + 3)}{3} = 4(n+1) = 36 \]

**Theorem 3** Let \( \langle x_n \rangle \) be an infinite sequence with zero-column \( \langle c_0, c_1, c_2, \ldots, c_p, 0, 0, \ldots \rangle \). Then

\[ \sum_{k=0}^{n} x_k = c_0 \binom{n+1}{1} + c_1 \binom{n+1}{2} + \ldots + c_p \binom{n+1}{p+1} \]

**Proof:** We know from Prop. 9 from the Binomial Coeff. Chapter.

\[ \sum_{k=0}^{n} \binom{n}{k} = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n} = \binom{n+1}{n+1} \]

So

\[ \sum_{k=0}^{n} x_k = \sum_{k=0}^{n} \left[ c_0 \binom{k}{0} + c_1 \binom{k}{1} + \ldots + c_p \binom{k}{p} \right] \]

\[ = c_0 \sum_{k=0}^{n} \binom{k}{0} + c_1 \sum_{k=0}^{n} \binom{k}{1} + \ldots + c_p \sum_{k=0}^{n} \binom{k}{p} \]

\[ = c_0 \binom{n+1}{1} + c_1 \binom{n+1}{2} + \ldots + c_p \binom{n+1}{p+1} \]

**Ex 5** Let \( \langle x_n \rangle = \langle n^4 \rangle \). Find \( \sum_{k=0}^{n} x_k \).

**Sol.** The zero-column of \( \langle x_n \rangle \) is \( \langle 0, 1, 4, 16, 36, 84, 0, 0, \ldots \rangle \)

\[ \sum_{k=0}^{n} n^4 = 0 \cdot \binom{n+1}{1} + 1 \cdot \binom{n+1}{2} + 4 \cdot \binom{n+1}{3} + 36 \cdot \binom{n+1}{4} + 84 \cdot \binom{n+1}{5} \]

\[ = \ldots = \frac{n(n+1)(2n+1)(3n^2 + 3n - 1)}{30} \]

\[ = \frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30}. \]
§2. The Stirling Numbers of the First & Second kinds

The numbers that occur in the zero column of the sequence \( \langle n^0 \rangle \) have combinatorial significance.

\[
\begin{align*}
\langle n^0 \rangle & : 1, 1, 1, 1, 1, 1, \ldots \\
\langle \Delta (n^0) \rangle & : 0, 0, 0, 0, 0, 0, \ldots \\
\langle n^1 \rangle & : 0, 1, 2, 3, 4, 5, \ldots \\
\langle \Delta (n^1) \rangle & : 1, 1, 1, 1, 1, 1, \ldots \\
\langle \Delta^2 (n^1) \rangle & : 0, 0, 0, 0, 0, 0, \ldots \\
\langle n^2 \rangle & : 0, 1, 4, 9, 16, 25, 36, \ldots \\
\langle \Delta (n^2) \rangle & : 1, 3, 5, 7, 9, 11, \ldots \\
\langle \Delta^2 (n^2) \rangle & : 2, 2, 2, 2, 2, \ldots \\
\langle \Delta^3 (n^2) \rangle & : 0, 0, 0, 0, 0, \ldots \\
\langle n^3 \rangle & : 0, 1, 8, 27, 64, 125, 216, \ldots \\
\langle \Delta (n^3) \rangle & : 1, 7, 19, 37, 61, 91, \ldots \\
\langle \Delta^2 (n^3) \rangle & : 6, 12, 18, 24, 30, \ldots \\
\langle \Delta^3 (n^3) \rangle & : 6, 6, 6, 6, \ldots \\
\langle \Delta^4 (n^3) \rangle & : 0, 0, 0, 0, \ldots \\
\langle n^4 \rangle & : 0, 1, 16, 81, 256, 625, 1296, \ldots \\
\langle \Delta (n^4) \rangle & : 1, 15, 65, 175, 369, 671, \ldots \\
\langle \Delta^2 (n^4) \rangle & : 14, 50, 110, 194, 302, \ldots \\
\langle \Delta^3 (n^4) \rangle & : 36, 60, 84, 108, \ldots \\
\langle \Delta^4 (n^4) \rangle & : 24, 24, 24, \ldots \\
\langle \Delta^5 (n^4) \rangle & : 0, 0, 0, \ldots \\
\end{align*}
\]

\[\Delta^k(n^0) = k! \{^p_k\} \text{ where } \{^p_k\} \text{ is the Stirling coeff. of the 2nd kind}\]
Def.: For each \( k \in \mathbb{N} \), we define the polynomial \([n]_k\) in \( n \) of degree \( k \) by \([n]_k = n(n-1)\ldots(n-k+1)\). So \([n]_0 = 1\), \([n]_1 = n\), \([n]_2 = n(n-1)\), and \([n]_k = \frac{n!}{(n-k)!}\).

Def.: For each \( k, p \in \mathbb{N} \), we define the Stirling numbers (or coefficients) of the second kind as the unique numbers \( \{P\}_k \) such that
\[
{n}^p = \sum_{k=0}^{p} \{P\}_k [n]_k.
\]

Note: Recall that \( \binom{p}{k} \) were the unique numbers such that \( (1+n)^p = \sum_{k=0}^{p} \binom{p}{k} n^k \). So there is a certain amount of similarity between \( \{P\}_k \) and \( \binom{p}{k} \).

Ex. 1: From the zero-column of \( \langle n \rangle_{\text{new}} \) and Prop. 2, we know that
\[
{n}^4 = 0 \cdot \binom{n}{0} + 1 \cdot \binom{n}{1} + 14 \cdot \binom{n}{2} + 36 \cdot \binom{n}{3} + 24 \cdot \binom{n}{4}.
\]
\[
= 0 \cdot [n]_0 + 1 \cdot [n]_1 + 14 \cdot [n]_2 + 36 \cdot [n]_3 + 24 \cdot [n]_4.
\]
\[
= 0 \cdot [n]_0 + 1 \cdot [n]_1 + 7 \cdot [n]_2 + 6 \cdot [n]_3 + 1 \cdot [n]_4.
\]
So \( \{4\}_0 = 0 \), \( \{4\}_1 = 1 \), \( \{4\}_2 = 7 \), \( \{4\}_3 = 6 \), and \( \{4\}_4 = 1 \).

Note: We can say \( \{4\}_k = 0 \) for \( k > 4 \), since they do not appear.

Prop. 4: For any \( k, p \in \mathbb{Z}^+ \) with \( 1 \leq k \leq p-1 \),
\[
\{P\}_k = \{P-1\}_k + k \cdot \{P-1\}_k \quad \text{(Stirling's Second Identity)}
\]

Note: This is very similar to the Pascal's identity,
\[
\binom{p}{k} = \binom{p-1}{k-1} + \binom{p-1}{k}.
\]
Proof: We have \( n^p = \sum_{k=0}^{p} \binom{p}{k} \cdot [n]_k \). Thus

\[
\begin{align*}
n^{p-1} &= \sum_{k=0}^{p-1} \binom{p-1}{k} \cdot [n]_k. \quad \text{Thus} \\
n^p &= n \cdot n^{p-1} = \sum_{k=0}^{p-1} \binom{p-1}{k} \cdot [n]_k. \quad n \\
&= \sum_{k=0}^{p-1} \binom{p-1}{k} \cdot [n]_k \cdot (n-k) + k. \\
&= \sum_{k=0}^{p-1} \binom{p-1}{k} \cdot [n]_k + k \cdot \sum_{k=0}^{p-1} \binom{p-1}{k} \cdot [n]_k \\
&= \sum_{k=0}^{p-1} \binom{p-1}{k} \cdot [n]_k + 0 + \sum_{k=1}^{p-1} k \cdot \binom{p-1}{k} \cdot [n]_k \\
&= 0 + \sum_{k=1}^{p-1} \left( \binom{p-1}{k-1} + k \cdot \binom{p-1}{k} \right) \cdot [n]_k + \binom{p-1}{0} [n]_p \cdot (**) \\
\end{align*}
\]

Comparing the coefficients of \((*)\) & \((***)\) we get

\[
\binom{p-1}{k} = \binom{p-1}{k-1} + k \cdot \binom{p-1}{k}, \quad \text{for} \quad 1 \leq k \leq p-1
\]

and \(\binom{p}{0} = \binom{p-1}{p-1}\) and \(\binom{p}{0} = 0^p\) for \(p \in \mathbb{N}\). \(\square\)

Ex. 2

(a) \(n^0 = 1 \cdot [n]_0 \quad \& \quad n^0 = \{0\} \cdot [n]_0 \Rightarrow \{0\} = 1 = 0^0\).

(b) \(n^1 = 0 \cdot [n]_0 + 1 \cdot [n]_1 \Rightarrow \{1\}_0 = 0 \quad \& \quad \{1\}_1 = 1\).

(c) \(n^2 = 0 \cdot [n]_0 + 1 \cdot [n]_1 + 1 \cdot [n]_2 \Rightarrow \{2\}_0 = 0, \quad \{2\}_1 = 1, \quad \varphi \{2\}_2 = 1\).

(d) \(n^3 = 0 \cdot [n]_0 + 1 \cdot [n]_1 + 3 \cdot [n]_2 + 1 \cdot [n]_3 \Rightarrow \)

\[
\begin{align*}
\{3\}_0 &= 0, \quad \{3\}_1 = 1, \quad \{3\}_2 = 3, \quad \{3\}_3 = 1.
\end{align*}
\]

Using Stirling's Second identity, we can compute the values of \(\varphi^3\) for all \(p, k \in \mathbb{N}\) much more easily.
is a non-negative integer.

Note: if we fill out the table as shown above, we can see that for all $k \geq 0$, $S_{k} = S_{k-1} + k \cdot S_{k-1}$. Hence, we have the following identity:

$$S_{k} = \sum_{i=0}^{k} \binom{k}{i} S_{i}$$

31 + 3(90) = 289 = 99 + 3 \cdot 86

We now turn to the Stirling numbers of the first kind.

Def. We define the Stirling numbers of the first kind as

$$[p] = \left\{ \begin{array}{ll} 0 & \text{if } p < 0 \\ 1 & \text{if } p = 0 \\ \frac{p!}{k!} & \text{if } p = k \\ 0 & \text{otherwise} \end{array} \right.$$
\[ [n]_q = n(n-1)(n-2)(n-3) = n(n^2-3n+2)(n-3) = n(n^3-6n^2+11n-6) = n^4-6n^3+11n^2-6n+1. \]

So \([4]_0 = 0, \quad [4]_1 = -6/(-1)^{4-1} = 6, \quad [4]_2 = 11/(-1)^{4-2} = 11, \]
\([4]_3 = -6/(-1)^{4-3} = 6, \] and \([4]_4 = 1/(-1)^{4-4} = 1. \]

**Prop. 5** For any \(k, p \in \mathbb{Z}^+\) with \(1 \leq k \leq p-1,\)
\[ [p]_k = [p-1]_{k-1} + (p-1) \cdot [p-1]_k. \quad \text{(Stirling’s First Identity)} \]

**Proof:** We know that \([n]_p = \sum_{k=0}^{p} (-1)^{p-k} \cdot [p]_k \cdot n^k \quad \text{(*)} \]

So \([n]_{p-1} = \sum_{k=0}^{p-1} (-1)^{p-1-k} \cdot [p-1]_k \cdot n^k. \quad \text{Thus} \]

\[ [n]_p = [n]_{p-1} \cdot \{n-(p-1)\} = \sum_{k=0}^{p-1} (-1)^{p-1-k} \cdot [p-1]_k \cdot n^k \cdot \{n-(p-1)\} = \sum_{k=0}^{p-1} (-1)^{p-1-k} \cdot [p-1]_k \cdot n^k \cdot (p-1) \]

\[ = \sum_{k=0}^{p-1} (-1)^{p-k} \cdot [p-1]_k \cdot n^k + \sum_{k=0}^{p-1} (-1)^{p-k} \cdot [p-1]_k \cdot n^k \cdot (p-1). \]

Comparing the coefficients of (\(*)\) & (\(**)\) we get
\[ [p]_k = [p-1]_{k-1} + (p-1) \cdot [p-1]_k \quad \text{for } 1 \leq k \leq p-1. \]

Also \([p]_p = [p-1]_0 \quad \text{&} \quad [p]_0 = 0 \quad \text{for } p \in \mathbb{Z}^+. \]
Ex 4 a) \( [n]_0 = 1 \) & \( [n]_0 = (-1)^{0-0} [1] n^0 = 1 \Rightarrow \binom{[1]}{0} = 1 \\
(b) \ [n]_1 = n \quad \& \quad [n]_1 = (-1)^{1-0} [1] n^0 + (-1)^{-1} [1] n^1 \\
\Rightarrow \binom{[2]}{1} = 0 \quad \& \quad \binom{[1]}{1} = 1 \\
(c) \ [n]_2 = n(n-1) = n^2 - n + 0 \cdot n^0 \\
\Rightarrow \binom{[3]}{0} = 0, \binom{[2]}{1} = 1 \quad \text{and} \quad \binom{[2]}{2} = 1. \\
(d) \ [n]_3 = n(n-1)(n-2) = n(n^2 - 3n + 2) = n^3 - 3n^2 + 2n + 0 \cdot n^1 \\
\Rightarrow \binom{[4]}{0} = 0, \binom{[3]}{1} = 2, \binom{[3]}{2} = 3, \binom{[3]}{3} = 1.

By using Sterling's First identity, we can compute the values of \( \binom{p}{k} \) for all \( p, k \in \mathbb{N} \) much more easily.

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<th>3</th>
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</tr>
</tbody>
</table>

\[ 50 + 5 (35) = \binom{4}{0} + (6-1) \binom{3}{3} = \binom{6}{0} = 225 \]

Observe also that \( \binom{p}{p-1} = \binom{p}{2} \) for all \( p \in \mathbb{Z}^+ \).

Note: If we fill out the table as shown above, we can see that for all \( k, p \in \mathbb{Z}^+ \), \( \binom{p}{k} = \binom{p-1}{k-1} + (p-1) \binom{p-1}{k-1} \)
§3. Combinatorial significance of the Stirling Numbers.

Our aim in this section is to show that the Stirling numbers of the Second & First kind have combinatorial meanings just as the Binomial coefficients \( \binom{n}{k} \) is also the number of \( k \)-subsets of \( \{1,2,\ldots,p\} \).

Def. Let \( k, p \in \mathbb{N} \) with \( k \leq p \). We define \( S(p,k) \) by
\[
S(p,k) = \text{no. of partitions of } \{1,2,\ldots,p\} \text{ into } k \text{ parts}.
\]

Recall that a partition of \( \{1,2,\ldots,p\} \) is a collection of disjoint, non-empty subsets \( \{A_i : i \in I\} \) of \( \{1,2,\ldots,p\} \) such that
\[
\bigcup_{i \in I} A_i = \{1,2,\ldots,p\}. \text{ If } |I| = k, \text{ then we say that the partition has } k \text{ parts.}
\]

Ex. 1 (a) \( \{1,2,3,\{3,4\}\} \) is a partition of \( \{1,2,3,4\} \) with 2 parts.

(b) \( \{\{1,2\}, \{3\}, \{4\}\} \) and \( \{\{1,2,3\}, \{3,4\}\} \) are both partitions of \( \{1,2,3,4\} \) with 3 parts.

Ex. 2 Let \( p = 4 \). Then \( \{\{1,3,\{2,4\}\}, \{\{2,3,4\}\}, \{\{2,3,4\}\}\} \), \( \{\{2,3,4\}\}, \{\{1,3,4\}\}, \{\{2,3,4\}\}, \{\{2,3,4\}\}\) and \( \{\{1,4\}, \{2,3\}\} \) are all the possible partitions of \( \{1,2,3,4\} \) into 2 parts. \( \therefore S(4,2) = 7 \).

Note: \( S(p, p) = 1 \) for \( p \in \mathbb{N} \) & \( S(p, 0) = 0 \) for \( p \in \mathbb{Z}^+ \) because \( \{\{1,2,3,\ldots,p\}\} \) is the only partition of \( \{1,2,\ldots,p\} \) into \( p \) parts.
Prop. 6: For each \( k, p \in \mathbb{Z}^+ \) with \( 1 \leq k < p \),
\[
S'(p, k) = S'(p-1, k-1) + k \cdot S'(p-1, k).
\]

Proof: Let \( A = \) set of all partitions of \( \{1, 2, \ldots, p\} \) into \( k \) parts,
\( B = \) set of partitions in \( A \) with \( p \) in a part by itself, and
\( C = \) set of partitions in \( A \) with \( p \) not in a part by itself.
Then \( B \cap C = \emptyset \) and \( A = B \cup C \).
So
\[
|A| = |B| + |C|.
\]
Now if we remove the part \( Sp^3 \) from each of
the partitions in \( B \), we will get a partition
of \( \{1, 2, \ldots, p-1\} \) into \( k-1 \) parts. And if we add
\( Sp^3 \) to any partition of \( \{1, 2, \ldots, p-1\} \) into \( k-1 \) parts,
then we will get a partition of \( B \).
So
\[
|B| = S'(p-1, k-1).
\]
Also if we remove \( p \) from its part in a
partition of \( C \), we will get a partition of
of \( \{1, 2, \ldots, p-1\} \) into \( k \) parts. And if we
add \( p \) to each part, in turns, to a partition
of \( \{1, 2, \ldots, p-1\} \) into \( k \) parts, we will get
\( k \) partitions of \( C \). So
\[
|C| = k \cdot S(p-1, k).
\]
Thus
\[
S'(p, k) = |A| = |B| + |C| = S'(p-1, k-1) + k \cdot S(p-1, k).
\]

Corollary 7: For each \( k, p \in \mathbb{N} \),
\[
S(p, k) = \{kp\}.
\]

Proof: \( S(p, k) \) & \( \{kp\} \) satisfy the same recurrence
equation with the same boundary conditions.
Hence we must have \( S(p, k) = \{kp\} \).
Def. Let $S_p$ be the set of permutations of $\{1, 2, \ldots, p\}$.

We define the relation $\sim$ on $S_p$ as follows:

$\langle i_1, \ldots, i_p \rangle \sim \langle j_1, \ldots, j_p \rangle$ if we can find a non-negative integer $k$ such that

$\langle i_1, \ldots, i_p \rangle = \langle j_{k+1}, j_{k+2}, \ldots, j_p, j_1, \ldots, j_k \rangle$

Ex. 3. Let $\langle i_1, i_2, i_3, i_4 \rangle = (1, 3, 2, 4)$ and $\langle j_1, j_2, j_3, j_4 \rangle = (2, 4, 1, 3)$. Then $\langle i_1, i_2, i_3, i_4 \rangle = \langle j_3, j_4, j_1, j_2 \rangle$. So $\langle 1, 3, 2, 4 \rangle \sim \langle 2, 4, 1, 3 \rangle$.

Fact 1. The relation $\sim$ is an equivalence relation on $S_p$ and it partitions $S_p$ into $(p-1)!$ equivalence classes each with $p$ elements. Each equivalence class will be called a circular permutation.

Ex. 4. The equivalence classes of $S_3$ are

$\{\langle 1, 2, 3 \rangle, \langle 2, 3, 1 \rangle, \langle 3, 1, 2 \rangle\}$ and $\{\langle 1, 3, 2 \rangle, \langle 3, 2, 1 \rangle, \langle 2, 1, 3 \rangle\}$

Fact 2. Each of the $(p-1)!$ equivalence classes of $S_p$ have an element which begins with "1". We shall use this permutation to represent the equivalence class from which it came. This permutation beginning with a "1" will also be called the representative of the circular permutation.

Ex. 5. The representative of the circular permutations of $\{1, 2, 3, 4\}$ are

$\langle 1, 2, 3, 4 \rangle, \langle 1, 2, 4, 3 \rangle, \langle 1, 3, 2, 4 \rangle, \langle 1, 3, 4, 2 \rangle$

$\langle 1, 4, 2, 3 \rangle, \langle 1, 4, 3, 2 \rangle$

So $\{1, 2, 3, 4\}$ has 6 circular permutations.
Note: If $A$ is any set, we usually pick a fixed element (usually the smallest element) to anchor the circular permutations. So the circular permutations of $\{2,3,5\}$ will be $\langle 2,3,5 \rangle$ and $\langle 2,5,3 \rangle$.

Here 2 serves as the anchor.

Def. Let $k, p \in \mathbb{N}$ with $k \leq p$. We define $s(p,k)$ by

\[ s(p,k) = \text{no. of arrangements of } 1,2,3,\ldots,p \text{ into } k \text{ non-empty circular partitions.} \]

In other words, $s(p,k)$ = no. of ways we can seat $1,2,3,\ldots,p$ at $k$ indistinguishable circular tables with no table being empty.

Ex. 6. Let us count the number of different ways we can seat $\{1,2,3,4\}$ at 2 indistinguishable tables.

\[
\begin{align*}
\langle 1,2 \rangle & \& \langle 3,4 \rangle \\
\langle 1,3 \rangle & \& \langle 2,4 \rangle \\
\langle 1,4 \rangle & \& \langle 2,3 \rangle \\
\langle 2,3,4 \rangle & \& \langle 1 \rangle \\
\langle 1,3,2 \rangle & \& \langle 4 \rangle \\
\langle 1,4,2 \rangle & \& \langle 3 \rangle \\
\langle 1,4,3 \rangle & \& \langle 2 \rangle \\
\langle 2,4,3 \rangle & \& \langle 1 \rangle \\
\end{align*}
\]

So $s(4,2) = 11$.

Note: For each $p \in \mathbb{N}$, $s(p,0) = 0^p$ and $s(p,p) = 1$.

Remember that $0^0 = 1$ and $0^p = 0$ for $p > 0$. 

Prop 8: For each k, p \in \mathbb{Z}^+ with 1 \leq k \leq p-1,

\[ s(p, k) = s(p-1, k-1) + (p-1) \cdot s(p-1, k). \]

Proof:
Let \( A \) = set of all seating arrangements of \( \{1, 2, \ldots, p\} \) at \( k \) indistinguishable tables with no tables empty. Put \( B = \) set of seatings in \( A \) with \( p \) at a table by itself & \( B_0 = \) set of seatings in \( A \) with \( p \) not by itself at a table. Then \( B \cap B_0 = \emptyset \) & \( A = B \cup B_0 \). So \( |A| = |B| + |B_0| \).

Now if we remove the table with \( p \) from each seating in \( B \), then we will get a seating of \( \{1, 2, \ldots, p-1\} \) at \( k-1 \) non-empty tables. And if we put \( p \) at a new table to any seating of \( \{1, 2, \ldots, p-1\} \) at \( k-1 \) non-empty tables, we will get a seating of \( B \). So

\[ |B| = s(p-1, k-1). \]

Also if we remove \( p \) from its table in a seating of \( B \), we will get a seating of \( \{1, 2, 3, \ldots, p-1\} \) at \( k \) non-empty tables. And if we seat \( p \) to the left of each of \( 1, 2, \ldots, p-1 \); in turns, at the respective tables of a seating of \( \{1, 2, \ldots, p-1\} \) at \( k \) non-empty tables, we will get \( p-1 \) seatings of \( B_0 \). So \( |B_0| = (p-1) \cdot s(p-1, k) \). Hence

\[ s(p, k) = |A| = |B| + |B_0| = s(p-1, k-1) + (p-1) \cdot s(p-1, k). \]

Corollary 9: For each \( k, p \in \mathbb{N} \), \( s(p, k) = \binom{p}{k} \).

Proof: \( s(p, k) \) & \( \binom{p}{k} \) satisfies the same recurrence equation with the same boundary conditions. Hence we must have \( s(p, k) = \binom{p}{k} \).
§4. Partitions of a non-negative integer

Ex.1. In how many ways can the multiset \([4, 4, 0]\) be partitioned into non-empty sub-multisets?

Sol. First of all remember that a join of multisets is not the same as a union of sets. So for example \([2a, 1b] + [1a, 3b] = [3a, 4b]\). A partition of the multiset \(M\) is a collection of sub-multisets \([A_1, \ldots, A_k]\) such that \(M = A_1 + \cdots + A_k\).

Two collections \([A_1, \ldots, A_k]\) & \([B_1, \ldots, B_k]\) are considered the same if \([A_1, \ldots, A_k] = [B_1, \ldots, B_k]\) as multi-multisets. Now the partitions of \([4, 4, 0]\) are

\[
\begin{align*}
[4a] & \quad 4 = 4 \\
[3a, 1a] & \quad 4 = 3 + 1 \\
[2a, 2a] & \quad 4 = 2 + 2 \\
[2a, 1a, 1a] & \quad 4 = 2 + 1 + 1 \\
[1a, 1a, 1a, 1a] & \quad 4 = 1 + 1 + 1 + 1.
\end{align*}
\]

This problem is equivalent to the number of ways of writing 4 as a sum of positive integers as shown above on the right. So our answer is 5.

Notice also that there are 2 ways of partitioning \([4, 4, 0]\) into 2 sub-multisets.

Def. Let \(k, n \in \mathbb{N}\). We define \(p(n, k)\) to be the number of partitions of \([n, 1]\) into \(k\) non-empty parts.

We also define \(p(n)\) to be the total number of partitions of \([n, 1]\). So \(p(n) = p(n, 1) + \cdots + p(n, n)\).
Ex. 2. Find \( p(6,2) \) & \( p(n,0) \).

Sol. We have \( 6 = 5+1, \ 6 = 4+2, \ 6 = 3+3 \). So \( p(6,2) = 3 \). Also, \( 0 = \) empty sum of pos. int., and since there is only one way to write this \( p(0,0) = 1 = 0^0 \). And if \( n > 0 \), then \( n \) cannot be expressed as an empty sum of positive integers. So in this case \( p(n,0) = 0 = 0^0 \). Thus for any \( n \in \mathbb{N} \), \( p(n,0) = 0^n \).

Prop. 10. For any \( k, n \in \mathbb{N} \) with \( 1 \leq k \leq n \),

\[
p(n,k) = p(n-1, k-1) + p(n-k, k)
\]

Proof: Let \( A \) be the collection of all partitions of \( n \) into exactly \( k \) non-empty parts. Put

- \( B = \) collection of partitions of \( A \) with a part of size 1 &
- \( B' = \) collection of partitions of \( A \) with no part of size 1.

Then \( |A| = |B| + |B'| \).

Now if we remove a part of size 1 from a partition of \( B \), we will get a partition of \( n-1 \) into \( k-1 \) non-empty parts. And if we add a part of size 1 to a partition of \( n-1 \) into \( k-1 \) non-empty parts, we will get a partition of \( B \). Hence \( |B| = p(n-1, k-1) \).

Also if we remove a "1" from each of the parts of a partition of \( B \), we will get a partition of \( n-k \) into \( k \) non-empty parts. And if we add a "1" to each part of a partition of \( n-k \) into \( k \) non-empty parts, we will get a partition of \( B \).
Proof: Hence \( |\mathcal{A}| = p(n-k,k) \). So
\[
p(n,k) = |\mathcal{A}| = |\mathcal{B}| + |\mathcal{C}| = p(n-1,k-1) + p(n-k,k).
\]

Using Proposition 10, we can compute the values of \( p(n,k) \), for all \( k, n \in \mathbb{N} \).

Using Proposition 10, we can compute the values of \( p(n,k) \), for all \( k, n \in \mathbb{N} \).

Ex. 3 (a) \[ p(8-1, 3-1) + p(8-3,3) = p(8,3) \]
\[
\begin{array}{c}
3 \\
2
\end{array}
+ \begin{array}{c}
2 \\
2
\end{array} = \begin{array}{c}
5
\end{array}
\]

(6) \[ p(8-1, 4-1) + p(8-4,4) = p(8,4) \]
\[
4 + 1 = 5
\]

Ex. 4 Let us now turn our attention to the collection of all partitions of a \([n,1]\). Consider the partition
\[
5 \quad 3 \quad 1 \quad 1
\]

This diagram is called the Ferrer's diagram of the partition.
Ex. 4 If we interchange the rows & columns of the Ferrer's diagram of 5+3+1+1, we will get the conjugate partition 4+2+2+1+1.

Def. A partition of \( n \) is **self-conjugate** if it is the same as its conjugate.

\[ 3+2+2 \text{ is a self-conjugate partition of } 8 \]

Prop. 11 Let \( q(n,k) \) be the number of partitions of \( n \) in which the largest part is of size \( k \). Then

\[ q(n,k) = p(n,k) \]

Proof: Recall that \( p(n,k) \) is the number of partitions of \( n \) into \( k \) non-empty parts. Let

\( A = \) collection of all partitions of \( n \) into \( k \) non-empty parts,

\( B = \) collection of all partitions of \( n \) in which the largest part has size \( k \).

Then the conjugate of each partition of \( A \) is a partition of \( B \). And the conjugate of each partition of \( B \) is a partition of \( A \). So \( |A| = |B| \). Hence \( q(n,k) = p(n,k) \).

Corollary 12: \[ q(n,1) + q(n,2) + \cdots + q(n,k) = p(n,1) + p(n,2) + \cdots + p(n,k) \].
Ex. 5. Let \( A_3 \) = collection of partitions of 7 into 3 parts & 
\( B_3 \) = collection of partitions of 7 with the largest part having size 3. Then
\[
A_3 = \{ 5+1+1, 4+2+1, 3+3+1 \} \text{ and } \ 
B_3 = \{ 3+1+1+1+1, 3+2+1+1, 3+2+2 \}.
\]
So \( |A_3| = |B_3| \).

Def. Let \( A_{\text{dist}}(n) \) = collection of all partitions of \( n \) in which each part of the partition is of different sizes, & 
\( A_{\text{odd}}(n) \) = collection of all partitions of \( n \) in which each part is of odd size.

We define \( P_{\text{dist}}(n) = |A_{\text{dist}}(n)| \) & \( P_{\text{odd}}(n) = |A_{\text{odd}}(n)| \).

Ex. 6. \( A_{\text{dist}}(5) = \{ 5, 4+1, 3+2 \} \)
\( A_{\text{odd}}(5) = \{ 5, 3+1+1, 1+1+1+1 \} \)
Notice that \( |A_{\text{dist}}(5)| = |A_{\text{odd}}(5)| \).

Algorithm 1 (Distinct parts into Odd parts algorithm)

**INPUT:** A partition \( P \) of \( n \) into distinct parts.
**OUTPUT:** A partition \( Q \) of \( n \) into odd parts.

1. Let \( i = 0 \) and \( P_i \leftarrow P \).
2. If \( P_i \) has no part of even size, STOP;
   else split each even part of \( P_i \) to get \( P_{i+1} \).
3. Let \( i \leftarrow i+1 \) and go to step 2.

Ex. 7. Find odd partitions of 10 corresponding to
(a) \( 5+3+2 \) \hspace{1cm} (b) \( 6+4 \)

(a) \( P_0 = 5+3+2 \). 
\( P_1 = 5+3+1+1 \). This is our final answer.
Ex. 7(b) \[ P_0 = 6 + 4 \]
\[ P_1 = 3 + 3 + 2 + 2 \]
\[ P_2 = 3 + 3 + 1 + 1 + 1 + 1 \] ← This is our final ans.

Algorithm 2 (Odd parts into Distinct parts algorithm)

INPUT: A partition \( Q \) of \( n \) into odd parts

OUTPUT: A partition \( P \) of \( n \) into distinct parts.

1. Let \( i \leftarrow 0 \) and \( Q_i \leftarrow Q \).
2. If \( Q_i \) has no two parts of the same size, STOP;
   else group the parts of the same size in pairs
   (leave out 1 if there are an odd no. of parts of
   the same size) and union each pair to get a
   new partition \( Q_{i+1} \).
3. Let \( i \leftarrow i + 1 \) and go to step 2.

Ex. 8. Find the distinct partitions of 10 corresponding to
(a) \( 3 + 3 + 1 + 1 + 1 + 1 \)
(b) \( 5 + 1 + 1 + 1 + 1 + 1 \)

(a) \( Q_0 = (3+3) + (1+1) + (1+1) \)
\( Q_1 = 6 + (2+2) \)
\( Q_2 = 6 + 4 \) done

(b) \( Q_0 = 5 + (1+1) + (1+1) + 1 \)
\( Q_1 = 5 + (2+2) + 1 \)
\( Q_2 = 5 + 4 + 1 \) done.

Theorem 13: \( \text{Pdist}(n) = \text{Podd}(n) \).
(Sketch of the)

Proof: Let \( f: \text{A}_{\text{DIST}}(n) \rightarrow \text{A}_{\text{ODD}}(n) \) be defined by \( f(P) = \)
the unique partition of \( n \) produced by Algorithm 1.
Also let \( g: \text{A}_{\text{ODD}}(n) \rightarrow \text{A}_{\text{DIST}}(n) \) be defined by \( g(Q) = \)
the unique partition of \( n \) produced by Algorithm 2.
Then \( f \circ g = \text{identity function} \) & \( g \circ f = \text{identity function} \). So
\( f \) is a bijection. \( \therefore \) \( \text{Pdist}(n) = \left| \text{A}_{\text{DIST}}(n) \right| = \left| \text{A}_{\text{ODD}}(n) \right| = \text{Podd}(n) \).
Ex. 5. Let \( A_3 \) = collection of partitions of 7 into 3 parts &
\( B_3 \) = collection of partitions of 7 with largest part of size 3. Then
\[
A_3 = \{ 5+1+1, 4+2+1, 3+3+1 \}
\]
\[
B_3 = \{ 3+1+1+1, 3+2+1+1, 3+2+2 \}
\]
This verifies that \( |A_3| = |B_3| \).

Def. Let \( A_D(n) = \) collection of all partitions of \( n \) into parts which are all of different sizes, and
\( B_D(n) = \) collection of all partitions of \( n \) into parts which are all of odd sizes.
We define \( \text{Dist}(n) = |A_D(n)| \) & \( \text{Odd}(n) = |B_D(n)| \).

Ex. 6
\[
A_D(5) = \{ 5, 4+1, 3+2 \}
\]
\[
B_D(5) = \{ 1+1+1+1+1, 3+1+1, 5 \}
\]
\[= \{ 5, [1], 2, [1]+[1], [3], 1, [5] \} \]

Theorem 13. For any \( n \in \mathbb{N} \), \( \text{Dist}(n) = \text{Odd}(n) \).

Proof. Let us consider any partition of \( B_D(n) \). Then we can express that partition in the form
\[
[n] = x_1 \cdot [1] + x_2 \cdot [3] + x_5 \cdot [5] + \ldots
\]
as indicated in example 6. Now express each \( x_i \) as a binary numeral in reverse order,
\[
x_i = 2^{a_i} 2^{b_i} 2^{c_i} \ldots \text{ with } a_i < b_i < c_i < \ldots
\]
Then we will have
\[
n = 1 \cdot 2^{a_i} + 1 \cdot 2^{b_i} + 1 \cdot 2^{c_i} + \ldots \quad \text{(each term is)}
\]
\[+ 3 \cdot 2^{a_3} + 3 \cdot 2^{b_3} + 3 \cdot 2^{c_3} + \ldots \quad \text{(one part of } n \text{)}
\]
\[+ 5 \cdot 2^{a_5} + 5 \cdot 2^{b_5} + 5 \cdot 2^{c_5} + \ldots \quad + \ldots
\]
(C6 parts are listed followed by dots – but there could be less or more).
Proof. But this gives us a partition of \( n \) in which all parts are of different sizes, because

\[
2^a < 2^b < 2^c < \ldots \\
3 \cdot 2^a < 3 \cdot 2^b < 3 \cdot 2^c < \ldots \\
5 \cdot 2^a < 5 \cdot 2^b < 5 \cdot 2^c < \ldots
\]

since \( a_i < b_i < c_i < \ldots \) for each odd \( i \) and

\[
2^a, 3 \cdot 2^a, 5 \cdot 2^a, \ldots \text{ are always different.}
\]

So each partition of \( B_0(n) \) corresponds to a partition of \( A_0(n) \).

Now consider any partition of \( A_0(n) \).
Write it as \( n = y_1 + y_2 + y_3 + \ldots \) and express each \( y_i \) in the form \( 2^a (2k+1) \).

Then by adding together the the portions with the same odd part, \( 2k+1 \), we will get

\[
[n] = \lambda_1[1] + \lambda_3[3] + \lambda_5[5] + \ldots
\]

which is a partition of \( B_0(n) \). Notice that

\[
\lambda_1 \text{ = coeff. of all the portions of the form } 2^a 1
\]

\[
\lambda_3 \text{ = coeff. of all the portions of the form } 2^a 3
\]

and so on. So each partition of \( A_0(n) \) corresponds to a partition of \( B_0(n) \). Hence

\[
\text{Pdist}(n) = |A_0(n)| = |B_0(n)| = \text{Podd}(n).
\]

\textbf{Example.} \( B_0(7) \) is:

\[
\{1+1+1+1+1+1, 1+1+1+1+3, 1+1+5, 7 \}
\]

\[
= \{2^0+2^0+2^0, 1, 2^1+2^0+2^0, 2^1+2^0+2^0, 2^0+1 \}
\]

\[
\sim \{1+2+4, 4+3, 2+5, 7\} = A_0(7).
\]

\( A_0(6) \) is:

\[
\{2+4, 1+2+3, 6, 1+5 \}
\]

\[
= \{2^0+2^1, 1, 2^0+2^0+2^0, 2^1+2^0+2^0 \}
\]

\[
\sim \{6, 1, 3+1+1, 3+3, 1+1+1+1+1 \}
\]

\[
\sim \{ 1+1+1+1+1, 1+1+1+3, 3+3, 1+5 \} = B_0(6).
\]
The Placement of balls into boxes.

Ex. 1
In how many ways can we distribute 4 balls into 2 boxes?
The answer depends on the kinds of balls and the kind of boxes. Possible labels for balls: a, b, c, d.
possible labels for boxes: 1st, 2nd.

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<th>EMPTY BOXES ALLOWED?</th>
<th>ANSWER</th>
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Fact:

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<th>k BOXES LABELLED?</th>
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<th>ANSWER</th>
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<td>YES</td>
<td>(p(n,k) + \ldots + p(n,0))</td>
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<td>YES</td>
<td>NO</td>
<td>(\binom{n-1}{k-1})</td>
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<tr>
<td>NO</td>
<td>YES</td>
<td>YES</td>
<td>(\binom{n}{k} + \binom{n}{k-1} + \ldots + \binom{n}{0})</td>
</tr>
<tr>
<td>YES</td>
<td>NO</td>
<td>NO</td>
<td>(k!\binom{n}{k})</td>
</tr>
<tr>
<td>YES</td>
<td>YES</td>
<td>NO</td>
<td>(k^n = \sum_{i=0}^{k} [k]_i \binom{n}{i})</td>
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