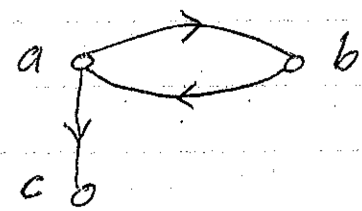
CHAPTER 1 – Basic concepts of Graph Theory

§1. Digraphs, graphs & other similar objects

Graph is mostly the study of graph and digraphs. The more fundamental object is the digraph (which is an abbreviation of directed graph).

Def A *digraph* is an ordered pair $G = \langle V, E \rangle$ where V is a non-empty set and E is an irreflexive relation on V.
(Recall that a *relation* on V is just a set of ordered pairs of elements of V. The relation E is irreflexive if for each $a \in V$, $\langle a, a \rangle \notin E$.)

Ex. 1 Let $V = \{a, b, c\}$ and $E = \{\langle a,b \rangle, \langle b,a \rangle, \langle a,c \rangle\}$. Then $G = \langle V, E \rangle$ is a digraph. If we represent the elements of V by small circles and an element $\langle a,b \rangle$ of E by a curve directed from a to b. So

$G =$ 

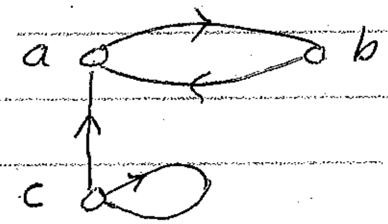
The elements of V are called *vertices* (or *nodes*) and the elements of E are called *directed edges* (or directed arcs). In order to get a more compact and natural description, we usually abbreviate the directed edge $\langle a,b \rangle$ by $\vec{ab}$. So we can rewrite G as

$$G = \langle \{a,b,c\}, \{\vec{ab}, \vec{ba}, \vec{ac}\} \rangle.$$

We usually use $V(G)$ & $E(G)$ to identify the set of vertices & the set of directed edges of a digraph G, when more than a single digraph is under consideration. We can generalize the concept of a digraph by relaxing the requirement that the relation E be irreflexive.

Def. A <u>pseudo-digraph</u> is an ordered pair $G = \langle V, E \rangle$ where V is a non-empty set and E is a relation on V.

Ex. 2 Let $V = \{a, b, c\}$ and $E = \{\langle a,b \rangle, \langle b,a \rangle, \langle c,a \rangle, \langle c,c \rangle\}$. Then $G = \langle V, E \rangle$ is a pseudo-digraph. We can also abbreviate the directed edges in E again to get $E = \{\vec{ab}, \vec{ba}, \vec{ca}, \vec{cc}\}$.

$G =$ 

A directed edge of the form $\langle c,c \rangle$ is called a <u>loop</u>. We can further generalize the concept of a digraph by allowing E to be a multi-set of ordered pairs of vertices.

Def. A <u>multi-digraph</u> is an ordered pair $G = \langle V, E \rangle$ where V is a non-empty set & E is a multi-set of ordered pairs of <u>distinct</u> vertices. A <u>multi-pseudo-graph</u> (or digraph-like object is an ordered pair $G = \langle V, E \rangle$ where V is a non-empty set and E is a multi-set of ordered pairs of vertices.

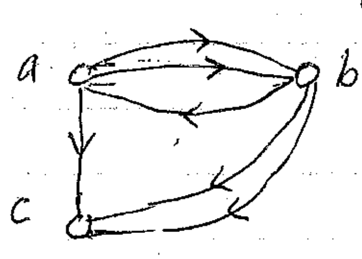
Recall that a <u>multi-set</u> is an ordered pair $\langle A, f \rangle$ where A is set and $f: A \to \mathbb{Z}^+$ is a function from A to $\mathbb{Z}^+$. Here $\mathbb{Z}^+$ is the set of positive integers. We usually write a multi-set using square brackets as $M = [f(a_1)\cdot a_1, f(a_2)\cdot a_2, \cdots, f(a_n)\cdot a_n]$ when $A = \{a_1, \dots, a_n\}$. We can also repeat the element a_i, $f(a_i)$ times. So

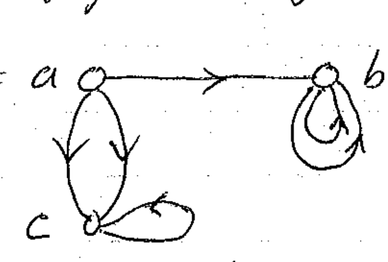
$$M = [3\cdot a, 2\cdot b] = [a, a, a, b, b]$$

is a multi-set with 5 elements. Note that the order of the elements of a multi-set are irrelevant just as they are in a set, but the number of times an element is present matters – unlike the case with with sets. So

$$[a, a, b] = [a, b, a] = [b, a, a],$$
$$\{a, a, b\} = \{a, b\}, \quad \text{and} \quad [a, a, b] \neq [a, b].$$

<u>Ex.3</u> Let $V = \{a, b, c\}$, $E_1 = [2\cdot\langle a,b\rangle, 1\cdot\langle b,a\rangle, 1\cdot\langle a,c\rangle, 2\cdot\langle b,c\rangle]$ and $E_2 = [1\cdot\langle a,b\rangle, 2\cdot\langle a,c\rangle, 2\cdot\langle b,b\rangle, 1\cdot\langle c,c\rangle]$. Then $G_1 = \langle V, E_1\rangle$ is a multi-digraph and $G_2 = \langle V, E_2\rangle$ is a multi-psedo-digraph (or digraph-like object)

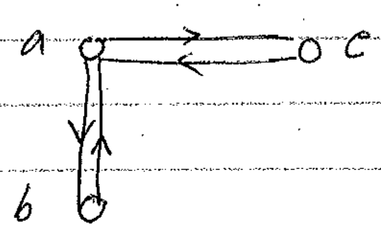
$G_1 =$ 

$G_2 =$ 

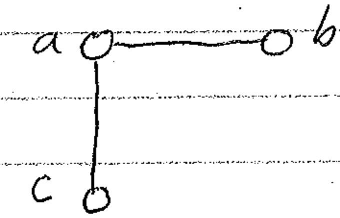
We can view the directed edges of a digraph as one-way streets. Now if we can travel in both directions on a street, then we can remove the one-way arrow on the street. Similarly, if $\langle a,c\rangle$ and $\langle c,a\rangle$ are both present in a digraph G, then we can view these two directed edges as a single (two-way) edge between a and b. If all the directed edges of G come in such pairs, then we will arrive at the concept of a graph. So a graph will be a special digraph

<u>Def.</u> A <u>graph</u> is an ordered pair $G=\langle V,E\rangle$ where V is a non-empty set and E is an irreflexive and symmetric relation on V.
(Recall that a relation E is symmetric if $\langle a,b\rangle \in E \Rightarrow \langle b,a\rangle \in E$.)

Ex. 4. Let $V=\{a,b,c\}$ and $E=\{\langle a,b\rangle, \langle b,a\rangle, \langle a,c\rangle, \langle c,a\rangle\}$
Then $G=\langle V,E\rangle$ is a graph.



G viewed as a digraph



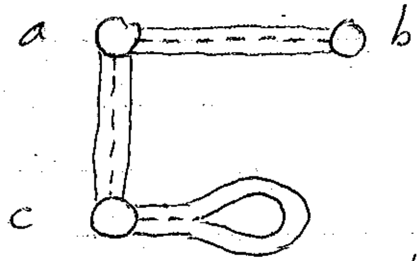
G viewed as a graph

We abbreviated the directed edge $\langle a,b\rangle$ as $\vec{ab}$ and we will abbreviate the pair of oppositely matched directed edges $\vec{ab}$ & $\vec{ba}$ by $\bar{ab}$ (or just simply as ab)

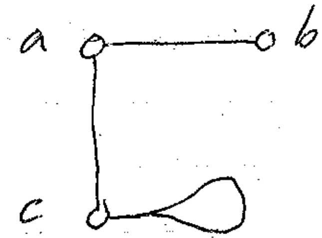
So $G=\langle\{a,b,c\}, \{\bar{ab}, \bar{ac}\}\rangle = \langle\{a,b,c\}, \{ab, ac\}\rangle$.

Def. A <u>pseudo-graph</u> is an ordered pair $G = \langle V, E \rangle$ where V is a non-empty set and E is a symmetric relation on V.

Ex 5. Let $V = \{a, b, c\}$ and $E = \{\langle a,b\rangle, \langle b,a\rangle, \langle a,c\rangle, \langle c,a\rangle, \langle c,c\rangle\}$
Then $G = \langle V, E \rangle$ is a pseudo-graph



G viewed as road system

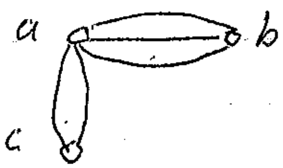


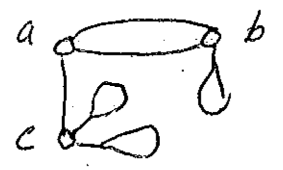
G viewed as a pseudo-graph

The edge $\langle c,c\rangle$ is still called a <u>loop</u>. We can generalize graphs by allowing E to be a multi-set

Def. A <u>multi-graph</u> is an ordered pair $G = \langle V, E \rangle$ where V is a non-empty set and E is a multi-set of oppositely matched ordered pairs of <u>distinct</u> vertices
A <u>multi-pseudo-graph</u> (or <u>graph-like object</u>) is an ordered pair $G = \langle V, E \rangle$ where V is a non-empty set and E is a multi-set of oppositely matched ordered pairs of distinct vertices, or of ordered pairs of identical vertices.

Ex. 6 Let $V = \{a, b, c\}$, $E = [2 \cdot \langle a,b\rangle, 2\langle b,a\rangle, 3\langle a,c\rangle, 3\langle c,a\rangle] = [2 \cdot \overline{ab}, 3 \cdot \overline{ac}]$
$E_2 = [2 \cdot \overline{ab}, 1 \cdot \overline{ac}, 1 \cdot \overline{bb}, 2 \cdot \overline{cc}]$. Then $G_1 = \langle V, E_1 \rangle$ is a multi-graph & $G_2 = \langle V, E_2 \rangle$ is a multi-pseudo-graph.

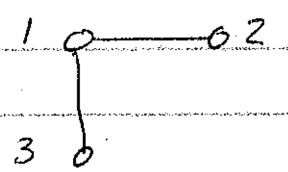
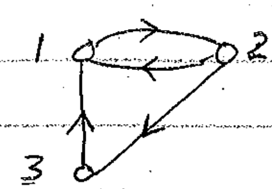
$G_1 =$ 

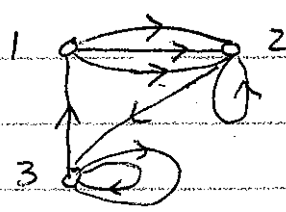
$G_2 =$ 

§2. <u>Representations of graphs & digraphs</u>:

A digraph-like object G, as defined, is a set-theoretical object. So we can call the definition, a set-theoretical representation of G. Unfortunately, this representation, even though it is very precise, is not very intuitive or easy to input into a computer program. For this reason we shall introduce two other kinds of representations.

<u>Def.</u> Let G be a digraph-like object with n vertices. Rename the vertices as $1, 2, 3, \ldots, n$ if necessary. We define the <u>adjacency</u> matrix of A_G of G by $A_G[i,j]$ = number of edges from i to j.

<u>Ex.1</u> Let G =  , H = 

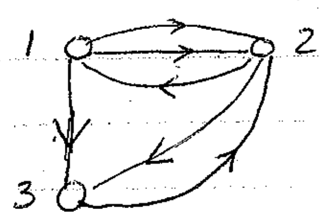
and K =  Then

$$A_G = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad A_H = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \ \& \ A_K = \begin{bmatrix} 0 & 3 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix}$$

From Ex. 1, we can readily see that the <u>adjacency matrix representation</u> is very compact and precise but it is not very intuitive or suggestive. All the same it is extremely useful.

Another way of representing graphs & digraphs by a matrix is by using the incidence relation of directed edges with their endpoints

<u>Def.</u> Let G be a multi-digraph with p vertices and q directed edges. Rename the vertices as $1, 2, 3, \ldots, n$ if necessary. We define the <u>incidence matrix</u> B_G of G as the $p \times q$ matrix with each of the q columns contain exactly one "1" and one "-1". Each column will correspond to one of the q directed edges with "1" in the row that corresponds to the initial vertex of the directed edge, and a "-1" in the row that corresponds to the terminal vertex of the directed edge.

<u>Ex 13</u> Let $G =$ 

. Then

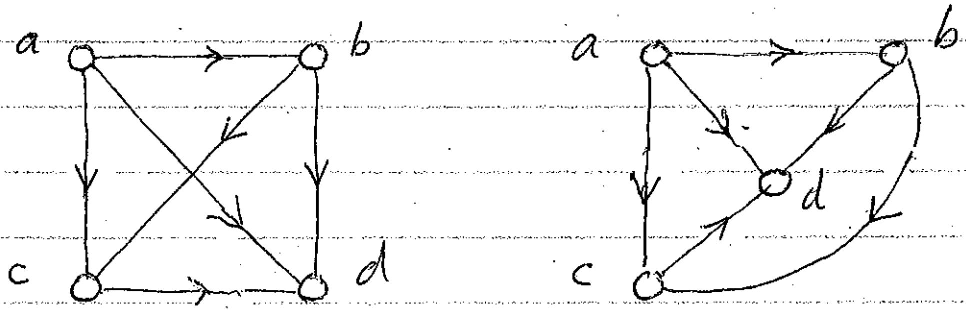
$$B_G = \begin{bmatrix} 1 & 1 & 1 & -1 & 0 & 0 \\ -1 & -1 & 0 & 1 & 1 & -1 \\ 0 & 0 & -1 & 0 & -1 & 1 \end{bmatrix}$$

In order to end up with a unique matrix B_G we also insist that the q edges must listed in the following order: First we list the $\langle i,j \rangle$ lexicographically. So all edges coming out of the vertex 1 will appear first with the terminating vertex determining the order. Then all edges coming out vertex 2, and so on.

Def. We can represent any digraph-like object G as a set of points in the plane $\mathbb{R}^2$ as follows. Each vertex of G will be a designated point in the plane. To high-light this point we will usually draw a small circle around it. An edge from the vertex a to the vertex b will be represented a <u>smooth</u>, directed continuous curve from the point representing a to the point representing b. We usually do not show the parts of the curves which are inside the small circles representing the vertices. This will give us a <u>geometric representation</u> of a graph-like object.

There are, unfortunately, many different geometric representations of a graph — but these representations are very intuitive & suggestive.

Ex.2 Let $V = \{a,b,c,d\}$ and $E = \{\vec{ab}, \vec{ac}, \vec{ad}, \vec{bc}, \vec{bd}, \vec{bd}\}$. Then two representations of $G = \langle V, E \rangle$ are shown below.



Even though these two representations look different, we can easily see that they have the same structure and features. The following definitions are inspired by the geometrical representation

Def. In a digraph-like object, a directed edge e is an ordered pair $\langle a,b \rangle$ and we usually abbreviate it as $\vec{ab}$. The vertex a is called the <u>initial endpoint</u> of e and b is called the <u>terminal endpoint</u> of. We say that e is <u>incident to</u> b and <u>incident from</u> a.

Def. In a digraph-like object, the <u>indegree</u> & <u>outdegree</u> of a vertex v are defined as follows.

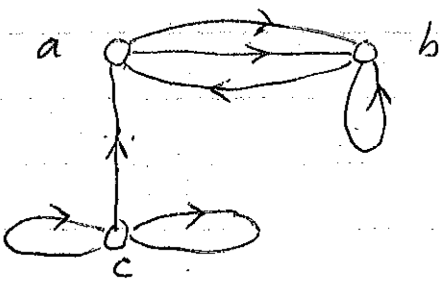
$\text{indeg}_G(v)$ = no. of times v served as a terminal endpoint.

$\text{outdeg}_G(v)$ = no. of times v served as an initial endpoint.

A vertex v in a digraph-like object is said to be <u>balanced</u> if $\text{indeg}_G(v) = \text{outdeg}_G(v)$. A digraph-like object is <u>balanced</u> if each vertex is balanced.

Ex. 3 Let $V = \{a, b, c\}$ & $E = \{\vec{ab}, \vec{ab}, \vec{ba}, \vec{ca}, \vec{bb}, \vec{cc}, \vec{cc}\}$

Then $G = \langle V, E \rangle$ is the digraph-like object below

$G =$ 

,

$$A_G = \begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{bmatrix} 0 & 2 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix} \end{array}$$

indeg $(a) = 2$, outdeg$(a) = 2$. So a is balanced.

indeg $(b) = 3$, outdeg $(b) = 2$

indeg $(c) = 2$, outdeg $(c) = 3$

Notice that indeg(b) = sum of column 2 of A_G and outdeg (b) = sum of row 2 of A_G. Also

sum of indegrees in G = sum of outdegrees in G = $7 = |E(G)|$

Prop. 1: In any digraph-like object $G = \langle V, E \rangle$
sum of indegrees in G = sum of outdegrees in G = $|E|$

Proof: Each directed edge $\vec{e}$ contributes one to the total indegree sum and one to the total outdegree sum. Since the indegrees & outdegrees are only generated by directed edges, it follows that
sum of indegrees in G = sum of outdegrees in G = $|E|$

Prop. 2: In any digraph-like object $G = \langle V, E \rangle$ with vertices $1, 2, \ldots, n$ and adjacency matrix A_G
(a) $\mathrm{indeg}(j)$ = sum of column j of A_G.
(b) $\mathrm{outdeg}(i)$ = sum of row i of A_G.

Proof: (a) $A_G[i,j]$ = number of edges from i to j.
So $\sum_{i=1}^{n} A_G[i,j]$ = sum of column j of A_G
= no. of times j served as a terminal endpt in G
= $\mathrm{indeg}(j)$.

(b) Similarly, it follows that
$\sum_{j=1}^{n} A_G[i,j]$ = sum of row i of A_G
= no. of times i served as an initial endpt in G
= $\mathrm{outdeg}(i)$.

Recall that graph-like object G is a special digraph-like object in which each vertex has oppositely matched directed edges coming in and going out of it. So all graph-like objects are balanced digraph-like objects, i.e., $\mathrm{indeg}_G(v) = \mathrm{outdeg}_G(v)$ for each v in $V(G)$.

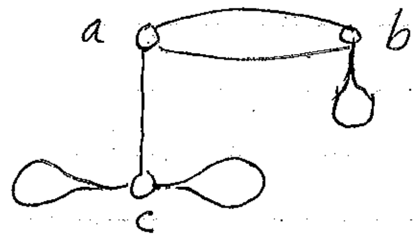
Def. In a graph-like object $G = \langle V, E \rangle$, the <u>degree</u> of a vertex v is defined by

$$\deg_G(v) = \text{indeg}_G(v) = \text{outdeg}_G(v)$$

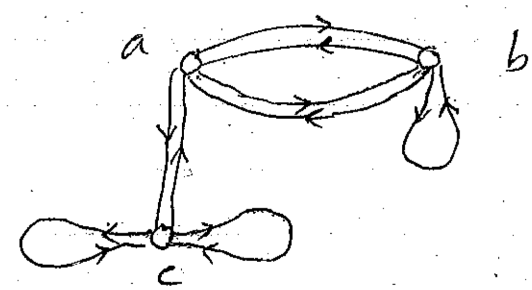
when G is viewed as a digraph-like object.

An <u>odd vertex</u> of G is any vertex with odd degree.
An <u>even vertex</u> of G is any vertex with even degree.

Ex 4 Let $V = \{a, b, c\}$ and $E = [\overline{ab}, \overline{ab}, \overline{ac}, \overline{bb}, \overline{cc}, \overline{cc}]$
Then $G = \langle V, E \rangle$ is a graph-like object.



G.



G as a digraph-like object

$\deg(a) = 3$, $\deg(b) = 3$, $\deg(c) = 3$.
Notice that each loop contributes only 1 to the degree of a vertex and thus only 1 to the total degree sum.

<u>Prop. 3</u>: Let $G = \langle V, E \rangle$ be any multi-graph. Then
(a) sum of the degrees in $G = 2|E|$
(b) G has an even number of odd vertices.

<u>Proof</u>: (a) First observe that each edge e contributes two to the total degree sum. Since there are $|E|$ edges, it follows that sum of degrees in $G = 2|E|$.

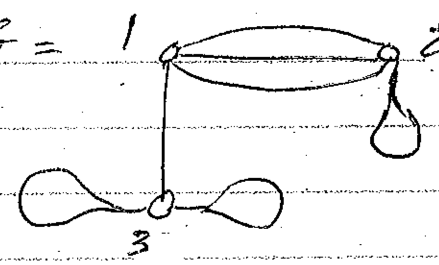
(b) We know that $2|E|$ = sum of the degrees in G
= sum of the even degrees + sum of the odd degrees
Since $2|E|$ and the the sum of the even degrees are both even, it follows that the sum of the odd degrees is even. Now this can only happen if there are an even number of odd vertices (because an odd number of odd numbers is odd).

Note: Proposition 3 is not true for pseudo-graphs or for multi-pseudo-graphs (graph-like objects) in general.

Prop. 4 If A_G is the adjacency matrix of a graph-like object with vertices $V = \{1, 2, 3, \ldots, n\}$, then
$\deg(i)$ = sum of column i = sum of row i.

Proof: This follows from Prop. 2 & the fact that $\deg(i) = \text{indeg}(i)$ when G is viewed as a digraph.

Ex. 5 Let G =



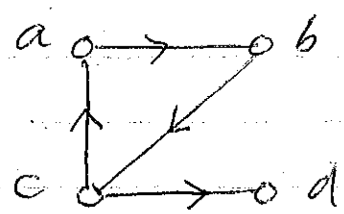
Then $A_G = \begin{bmatrix} 0 & 3 & 1 \\ 3 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$

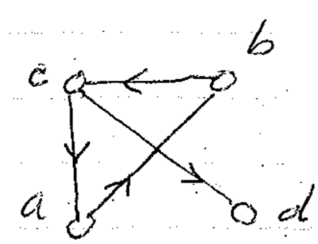
$\deg(1) = 0+3+1 = 4$ = sum of column 1
$\deg(2) = 3+1+0 = 4$ = sum of column 2
$\deg(3) = 1+0+3 = 3$ = sum of column 3.

Notice that sum of column i = sum of row i because column $i = (\text{row } i)^T$. This is because A_G is a symmetric matrix, i.e., $(A_G)^T = A_G$.

§3 Operations on graphs & other concepts

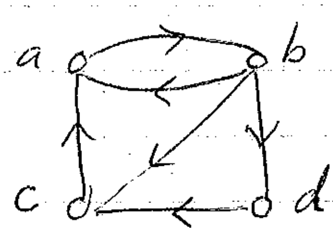
Def. Two digraphs (or graphs) G & H are *identical* if they have the same set-theoretical representation, i.e., if $V(G) = V(H)$ & $E(G) = E(H)$.

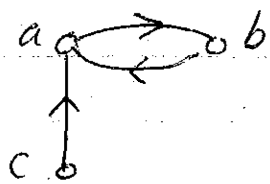
Ex.1 Let $G =$ 

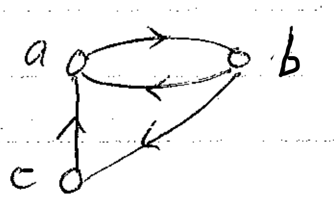
& $H =$ 

Then $G = \langle \{a,b,c,d\}, \{\langle a,b\rangle, \langle b,c\rangle, \langle c,a\rangle, \langle c,d\rangle\}$
and $H = \langle \{a,b,c,d\}, \{\langle a,b\rangle, \langle b,c\rangle, \langle c,a\rangle, \langle c,d\rangle\}$.
So $G \equiv H$.

Def. G is a *sub-digraph* of H if $V(G) \subseteq V(H)$ and $E(G) \subseteq E(H) \cap [V(G) \times V(G)]$. G is an *induced* sub-digraph of H if $V(G) \subseteq V(H)$ and $E(G) = E(H) \cap [V(G) \times V(G)]$. Sub-graphs and induced sub-graphs are defined similarly.

Ex.2 Let $H =$ 

, $G_1 =$ 

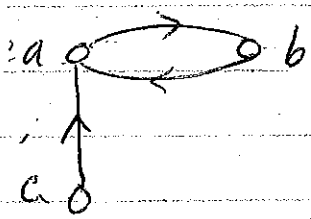
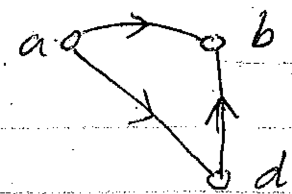
and $G_2 =$ 

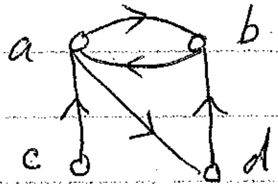
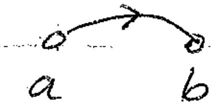
. Then G_1 is a sub-digraph of H & G_2 is an induced sub-digraph of H.

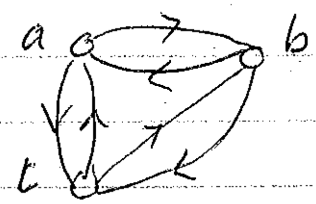
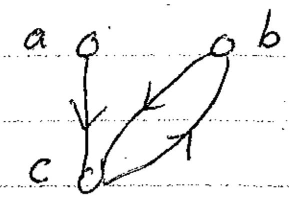
<u>Def.</u> Let $G = \langle V(G), E(G) \rangle$ and $H = \langle V(H), E(H) \rangle$ be digraphs. We define the <u>union</u> & <u>intersection</u> of G and H as follows.

$$V(G \cup H) = V(G) \cup V(H) \ \& \ E(G \cup H) = E(G) \cup E(H)$$
$$V(G \cap H) = V(G) \cap V(H) \ \& \ E(G \cap H) = E(G) \cap E(H).$$

We define the <u>completion</u> K_G of the digraph G by
$$V(K_G) = V(G) \ \& \ V(K_G) = V(G) \times V(G) - \{\langle x, x \rangle : x \in V(G)\}.$$
We define the <u>compliment</u> K^c of the digraph G by
$$V(G^c) = V(G) \ \& \ E(G^c) = E(K_G) - E(G).$$

<u>Ex.3</u> Let $G =$  & $H =$ 

Then $G \cup H =$ , $G \cap H =$ 

$K_G =$ , and $G^c =$ 

<u>Def.</u> We define the <u>null graph</u> on the vertices $V = \{1, 2, \dots, p\}$ by $N_p = \langle V, E \rangle$ where $E = \emptyset$. We define the <u>complete digraph</u> K_p on V by $K_p = N_p^c$. The <u>complete graph</u> K_p is the same as the complete digraph D_p, but each pair of oppositely matched directed edges is viewed as one edge in K_p.

<u>Prop. 5</u> Let $V = \{1, 2, 3, \ldots, p\}$. Then there are

(a) $2^{p(p-1)}$ non-identical digraphs on V

(b) $2^{p(p-1)/2}$ non-identical graphs on V

(c) 2^{p^2} non-identical pseudo-digraphs on V

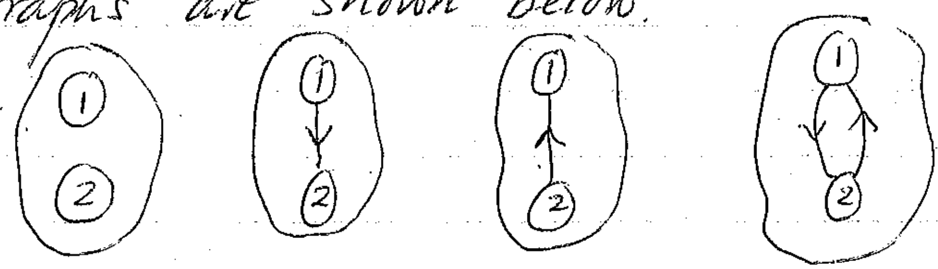
(d) $2^{p(p+1)/2}$ non-identical pseudo-graphs on V.

<u>Proof:</u>

(a) A directed edge consists of an ordered pair of <u>distinct</u> vertices of V. Since V has p vertices, there are thus $p(p-1)$ possible directed edges on V, Now each subset of these $p(p-1)$ directeds will give us a different digraph on V. Hence there are $2^{p(p-1)}$ non-identical digraphs on V.

(b) - (d): Do for H.W.

<u>Ex. 4</u> Let $V = \{1, 2\}$. Then the $2^{2(2-1)} = 4$ non-identical digraphs are shown below.



<u>Def.</u> Let G be graph. We define

$\Delta(G)$ to be the largest degree in G

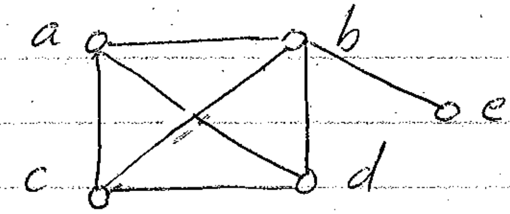
$\delta(G)$ to be the smallest degree in G

girth(G) to be the length of the smallest cycle in G

& circumference(G) to be the length of the largest cycle in G.

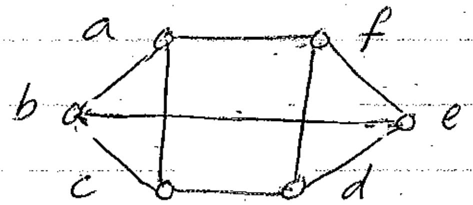
We also define the <u>degree sequence</u> of G to be the sequence of degrees of G in decreasing order. Finally a graph G is said to be <u>k-regular</u> if the degree of each vertex of G is k.

<u>Ex.5</u> Let $G =$



Then $\Delta(G) = 4$, $\delta(G) = 1$, girth $(G) = 3$, circumference $(G) = 4$ and the degree sequence of G is $\langle 4,3,3,3,1 \rangle$.

The graph H below is 3-regular.



<u>Def.</u> A sequence $\underset{\sim}{s} = \langle s_1, s_2, \dots, s_p \rangle$ is <u>graphical</u> if we can a graph G whose degree sequence is the same as the sequence $DEC(\underset{\sim}{s})$; where $DEC(\underset{\sim}{s}) =$ the terms of $\underset{\sim}{s}$ listed in decreasing order.

<u>Ex 6</u> The sequence $\langle 4,3,3,3,1 \rangle$ is graphical because this is the degree sequence of the graph in Ex.5. The sequence $\langle 4,3,3,2,1 \rangle$ is <u>not</u> graphical because the sum of the terms is odd and the sum of the terms of the degree sequence of a graph is even.

<u>Theorem</u> 6 (Graphical Sequence Theorem)
The decreasing sequence $d_1, d_2, \ldots, d_p$ of non-negative integers is graphical $\Leftrightarrow$ the sequence $d_2-1, d_3-1, \ldots, d_{d_1+1}-1, d_{d_1+2}, \ldots, d_p$ is graphical.

<u>Proof</u>: ($\Leftarrow$) Suppose $d_2-1, d_3-1, \ldots, d_{d_1+1}-1, d_{d_1+2}, \ldots, d_p$ is graphical. Then we can find a graph G' with vertices $v_2, \ldots, v_p$ such that

$$deg(v_i) = \begin{cases} d_i - 1 & \text{if } 2 \leq i \leq d_1+1 \\ d_i & \text{if } d_1+2 \leq i \leq p. \end{cases}$$

If we add a new vertex v_1 and edges between v_1 and $v_2, \ldots, v_{d_1+1}$ we will get a graph G with degree sequence $d_1, d_2, \ldots, d_p$. So the sequence $d_1, d_2, \ldots, d_p$ will be graphical.

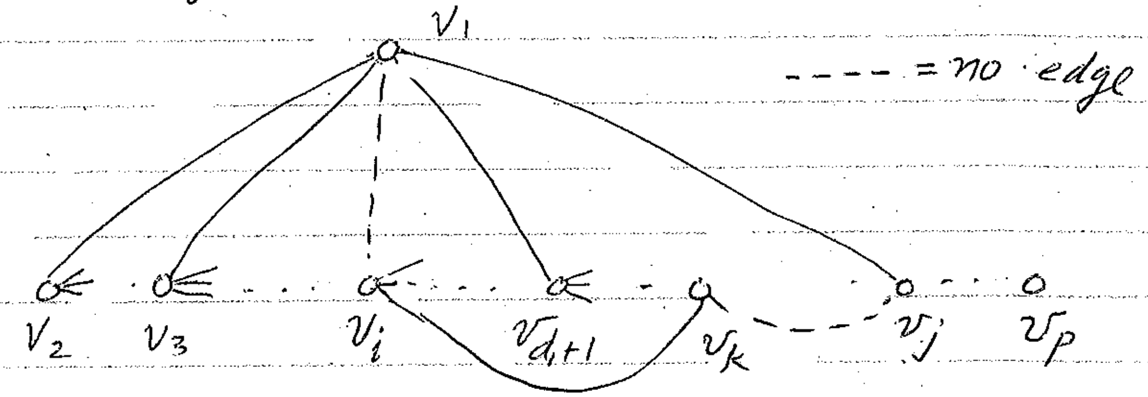
($\Rightarrow$) Suppose $d_1, d_2, \ldots, d_p$ is graphical. Then we can find a graph G with vertices $v_1, v_2, \ldots, v_p$ such that $deg(v_i) = d_i$. Now there may be many such graphs G. We <u>claim</u> that there is always one in which v_1 is adjacent to $v_2, \ldots, v_{d_1+1}$. Then all we have to do is to delete v_1 to get a graph G' with vertices $v_2, \ldots, v_p$ and

$$deg(v_i) = \begin{cases} d_i - 1 & \text{if } 2 \leq i \leq d_1+1 \\ d_i & \text{if } d_1+2 \leq i \leq d_p. \end{cases}$$

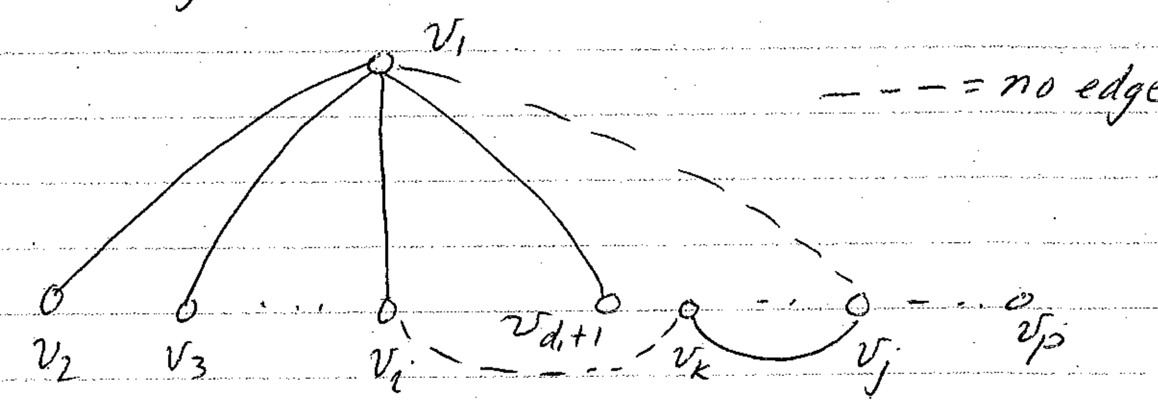
So $d_2-1, d_3-1, \ldots, d_{d_1+1}-1, d_{d_1+2}, \ldots, d_p$ will be graphical.

Let us now justify the claim. So suppose v_1 is not adjacent to the next d_1 largest degrees in G.

Then we can find vertices v_i and v_j such that $deg(v_i) > deg(v_j)$ while $v_1 v_i \notin E(G)$ & $v_1 v_j \in E(G)$.



Since $deg(v_i) > deg(v_j)$ there must be more edges out of v_i than out of v_j. So we can find a vertex v_k such that $v_i v_k \in E(G)$ & $v_j v_k \notin E(G)$. Now if let $G_1 = (G \cup \{v_1 v_i, v_j v_k\}) - \{v_1 v_j, v_i v_k\}$, then G_1 will have more of the next d_1 largest degrees adjacent to v_1 than G has.



If we keep repeating this process to G_1 we will get a graph G_2 which has even more of the next d_1 largest degrees adjacent to v_1 that G_1. In the end, we are sure to end up with a graph G_n which all the next d_1 largest degrees adjacent to v_1. So our claim is proved.

Qu: How can we tell if a sequence $\underset{\sim}{s}$ is graphical?

Algorithm 1 (Graphical Sequence Algorithm)

INPUT: A sequence $\underset{\sim}{s}$ of integers in decreasing order.

OUTPUT: YES, if $\underset{\sim}{s}$ is graphical; NO, if $\underset{\sim}{s}$ is not graphical.

1. If some integer in $\underset{\sim}{s}$ exceeds $|\underset{\sim}{s}|-1$, say NO & STOP. (Here $|\underset{\sim}{s}|$ = the length of $\underset{\sim}{s}$)
2. If each term of $\underset{\sim}{s}$ is zero, then say YES & STOP.
3. If some term of $\underset{\sim}{s}$ is negative then say NO & STOP.
4. Delete the first term s_1 from $\underset{\sim}{s}$ and then subtract 1 from the next s_1 terms from $\underset{\sim}{s}$ to get a new sequence $\underset{\sim}{s}'$
5. List the sequence $\underset{\sim}{s}'$ in decreasing order and rename it as $\underset{\sim}{s}$. Then go to step 2.

Ex. 7 Determine which of the following sequences are graphical? (a) $\langle 5,4,3,2,1,1\rangle$
(b) $\langle 5,3,3,2,2,1\rangle$ (c) $\langle 5,3,3,2,1\rangle$

(a) $5, \underline{4,3,2,1,1}$
$3, \underline{2,1,0,0}$
$1,0,-1,0$ $\boxed{NO}$

(b) $5, \underline{3,3,2,2,1}$
$2, \underline{2,1},1,0$
$1,0,1,0$
$1,\underline{1},0,0$ reorder
$0,0,0$ $\boxed{YES}$

(c) $|\underset{\sim}{s}| = |\langle 5,3,3,2,1\rangle| = 5$
Since $5 > |\underset{\sim}{s}|-1$, say $\boxed{NO}$

So $\langle 5,4,3,2,1,1\rangle$ and $\langle 5,3,3,2,1\rangle$ are not graphical.

Also $\langle 5,3,3,2,2,1\rangle$ is graphical.

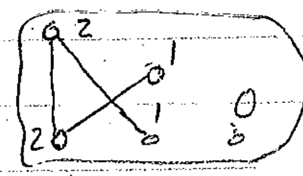
Algorithm 2 (Graph Recovery Algorithm)

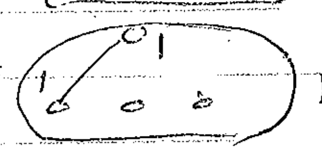
INPUT: A graphical sequence $\underset{\sim}{s}$ in decreasing order.

OUPUT: A graph with degree sequence $\underset{\sim}{s}$.

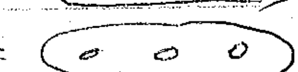
1. First perform the graphical sequence algorithm on the sequence $\underset{\sim}{s}$. We will end up with a sequence of k zeros. Let $G_1 \leftarrow N_k$ the null graph with k vertices and $i \leftarrow 1$
2. Each time step 4 of the graphical sequence algorithm is performed, let G_{i+1} be the graph obtained from G_i by adding a new vertex and edges from the new vertex to the vertices which corresponde to the terms which were decreased by 1.
3. In the end we will get a graph with degree sequence $\underset{\sim}{s}$. (<u>Note</u>: there may more than one such graph.)

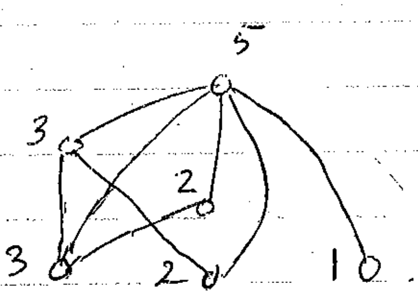
Ex. 8 $\quad 5, \underline{3,3,2,2,1}$

$2, \underline{2,1},1,0 \longrightarrow$ $G_3 =$ 

$1,0,1,0 \longrightarrow$ $G_2 =$ 

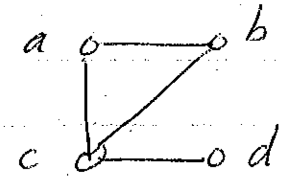
$1,\underline{1},0,0$ reorder

$0\ 0,0 \longrightarrow$ $G_1 =$ 

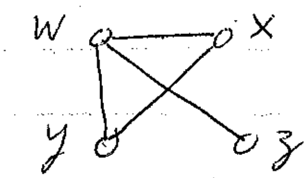
$G_4 =$ 

Def. Twodigraphs $G = \langle V(G), E(G) \rangle$ and $H = \langle V(H), E(H) \rangle$ are isomorphic (& we write $G \cong H$) if we can find a bijection $\alpha : V(G) \to V(H)$ such That $\langle u,v \rangle \in E(G) \Leftrightarrow \langle \alpha(u), \alpha(v) \rangle \in E(H)$. Two graphs G & H are isomorphic if we can find a bijection $\alpha : V(G) \to V(H)$ such that $uv \in E(G) \Leftrightarrow \alpha(u)\alpha(v) \in E(H)$.

Ex. 9 Let $G =$

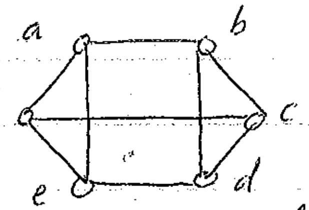


& $H =$

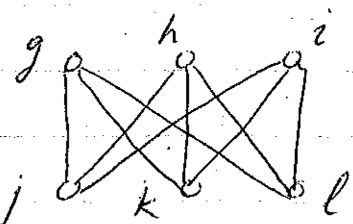


If we define $\alpha : V(G) \to V(H)$ by
$\alpha(a) = x$, $\alpha(b) = y$, $\alpha(c) = w$, and $\alpha(d) = z$
then we will see that $uv \in E(G) \Leftrightarrow \alpha(u)\alpha(v) \in E(H)$.
So $G \cong H$.

Ex. 10 Let $G =$



& $H =$



Then G has a cycle of length 3 but H has no cycle of length 3. Now if G was isomorphic to H then H would have also have a cycle of length 3. ($\because$ $\langle b, c, d, b \rangle$ is a cycle of length 3 & $\langle \alpha(b), \alpha(c), \alpha(d), \alpha(b) \rangle$ would have also formed a cycle of length 3.) Hence $G \not\cong H$.

No. of vertices	1	2	3	4	5	6
No. of non-isom. graphs	1	2	4	11	34	156
No. of non-isom. digraphs	1	3	16	218	9,608	1,540,944