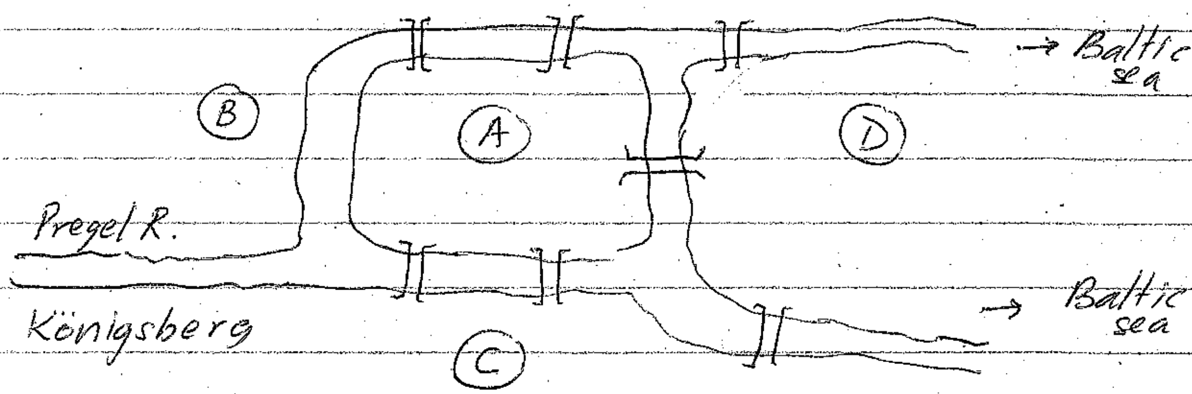
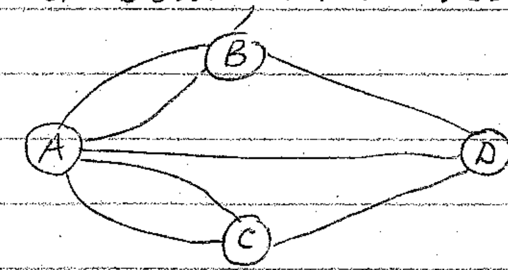
Ch.5 - Edge & vertex traversal problems

1. Euler circuits and Euler trails.

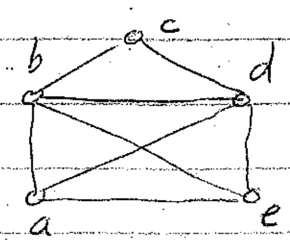
In the early 18th century there were seven bridges across various parts of the Pregel river when it passed through the city of Königsberg, Prussia.



The residents of this part of Königsberg amused themselves by asking if there is way of traversing each of the seven bridges exactly once. This question is equivalent to asking if there a trail, in the multigraph G below, which uses each edge exactly once.



<u>Def.</u> Let G be a graph-like object. An <u>Euler trail</u> of G is any trail in G which includes each edge of G exactly once. An <u>Euler circuit</u> of G is any <u>closed</u> Euler trail of G.



<u>Ex.1</u> a, b, c, d, e, b, d, a, e is an open Euler trail of the envelope graph on the right.

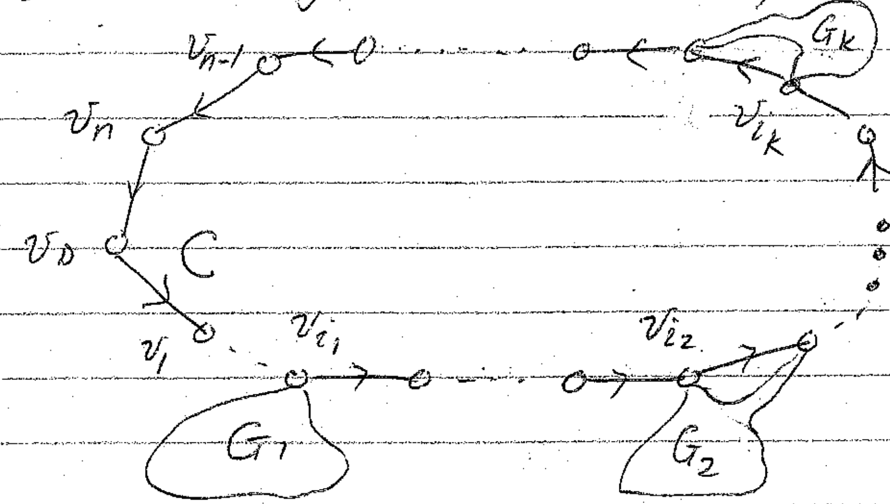
Theorem 1 (Euler's Circuit Theorem, 1735)
Let G be a connected multi-graph. Then G has an Euler circuit $\Leftrightarrow$ each vertex of G is of even degree.

Proof: ($\Rightarrow$) Suppose G has an Euler circuit — let's say it is $v_0, e_1, v_1, e_2, \ldots, v_q, e_q, v_0$ where $q = |E(G)|$. Then, starting at v_0, we traverse the circuit and delete each edge as we go along. Now observe that the degree of each vertex is reduced by 2 each time we pass through it. (In the beginning, the degree of v_0 will be reduced by 1; and in the end, it will also be reduced by 1.) When we finish traversing the circuit, each vertex will be of degree 0. Hence, at the start, the degree of each vertex must have been even.

($\Leftarrow$) Suppose each vertex of G is of even degree. We will show by induction on q that G has an Euler circuit.
Basis: If $q=0$, then G consists of a single vertex and the trivial walk, v_0, will be an Euler circuit of G.

Ind. Step. Suppose the result is true for all multi-graphs with $\leq q$ edges. Let G be any connected multi-graph with $q+1$ edges and all vertices of even degree. First we find a cycle in G, by starting at any vertex and keep going until a vertex is repeated for the first time. Let $v_0, v_1, \ldots, v_n, v_0$ be the vertex sequence of C. Now consider $G - E(C)$. If $G - E(C)$ has no edges, then C is an Euler circuit of G & we are done.

Otherwise, $G - E(C) = G_1 \cup G_2 \cup \ldots \cup G_k$ where the G_i's are disjoint connected components of $G - E(C)$.



Let v_{i_j} $(j = 1, \ldots, k)$ be the first vertex in C which is a member of G_j. Now each G_j is connected & has $\leq q$ edges and all vertices of even degree. So by the induction hypothesis, each G_j has an Euler circuit Q_j. We can get an Euler circuit of G as follows. Start at v_0, then go to v_{i_1} and traverse the Euler circuit Q_1, then go to v_{i_2} along C & traverse the Euler circuit Q_2, ..., then go to v_{i_k} along C & traverse the Euler circuit Q_k, and then return to v_0 along C. So if the result is true for all connected multi-graphs with $\leq q$ edges, it will be true for all connected multi-graphs with $q+1$ edges.

Conclusion: So by the Strong Principle of Mathematical Induction it follows that the result is true for all connected multi-graphs.

Corollary 2: Let G be a connected multi-graph. Then G has an open Euler trail $\Leftrightarrow$ exactly 2 vertices of G have odd degree.

Proof: ($\Rightarrow$) Suppose G has an open Euler trail. Let's say it is $v_0, e_1, v_1, e_2, \dots, v_{q-1}, e_q, v_q$. Put $G' = G \cup \{e\}$ where e is a new edge between v_q and v_0. Then $v_0, e_1, v_1, e_2, \dots, e_q, v_q, e, v_0$ will be an Euler circuit of G'. So each vertex of G' will be of even degree. Thus G must have exactly two vertices of odd degree, namely v_0 & v_q.

($\Leftarrow$) Suppose G has exactly two vertices v_0 & v_1 of odd degree. Let $G' = G \cup \{e\}$ where e is a new edge between the two odd vertices in G. Then G' is a connected multigraph with all vertices of even degree. So G' will have an Euler circuit Q, starting with v_0, e, v_1. Now if we remove both v_0 & e from Q, this will lead us to an open Euler trail of G from v_1 to v_0.

Def. Let G be a digraph-like object. A directed Euler trail of G is any directed trail of G which includes each directed edge of G exactly once. A directed Euler circuit is a closed directed Euler trail.

Theorem 3. Let G be a weakly-connected digraph-like object.

(a) Then G has a directed Euler circuit $\Leftrightarrow$ for each vertex v in G, $\text{indeg}(v) = \text{outdeg}(v)$, i.e., each v is balanced.

(b) Then G has an open directed Euler trail $\Leftrightarrow$ G has two vertices one with an extra outdeg. & the other with an extra in-degree and the rest of the vertices are balanced.

§2. Fleury's Algorithm & the Chinese Postman Problem.

<u>Algorithm 1</u> (<u>Fleury's Algorithm</u>, 1921)

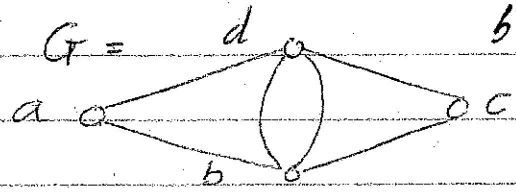
<u>INPUT</u>: A connected multi-graph G with at most two vertices of odd degree

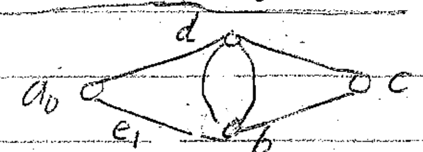
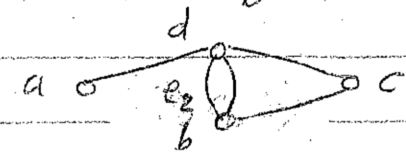
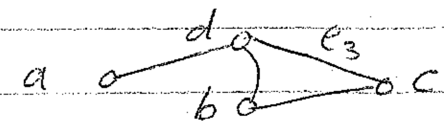
<u>OUTPUT</u>: An Euler trail of G.

1. Let $i \leftarrow 0$. If G has odd vertices, let v_0 be one of the two odd vertices; otherwise let v_0 be any vertex of G. Put $G_i \leftarrow G$ and $Q_i \leftarrow \langle v_0 \rangle$.
2. If v_i is a pendant vertex, let v_{i+1} be the only vertex adjacent to v_i and e_{i+1} be the edge from v_i to v_{i+1}, and let $G_{i+1} \leftarrow G_i - \{v_i\}$; otherwise, choose any edge e_{i+1} from v_i to an adjacent vertex v_{i+1} such that $G_i - \{e_{i+1}\}$ is still a connected graph, and let $G_{i+1} \leftarrow G_i - \{e_{i+1}\}$.
3. Let $Q_i \leftarrow Q_i \hat{\ } \langle e_{i+1}, v_{i+1} \rangle$ and $i \leftarrow i+1$. If $E(G_i) = \emptyset$, STOP; else, go to step 2.

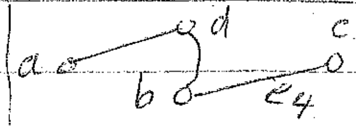
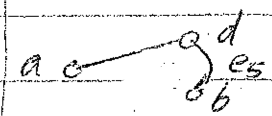
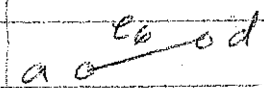
<u>Ex. 1</u> Find an Euler circuit of the multi-graph G below, by using Fleury's algorithm.

$G =$

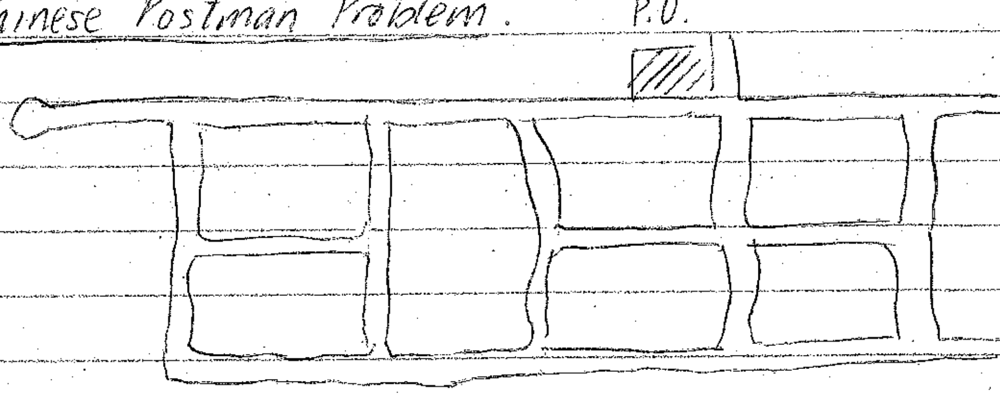


i	v_i	Q_i	G_i
0	a	$\langle a \rangle$	
1	b	$\langle a, e_1, b \rangle$	
2	d	$\langle a, e_1, b, e_2, d \rangle$	

Ex. 1

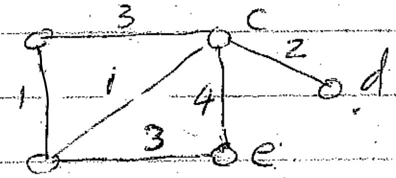
i	v_i	Q_i	G_i
3	c	$\langle a, e_1, b, e_2, d, e_3, c\rangle$	
4	b	$\langle a, e_1, b, e_2, d, e_3, c, e_4, b\rangle$	
5	d	$\langle a, e_1, b, e_2, d, e_3, c, e_4, b, e_5, d\rangle$	
6	a	$\langle a, e_1, b, e_2, d, e_3, c, e_4, b, e_5, d, e_6, a\rangle$	a STOP.

The Chinese Postman Problem.



We want a walk which starts and ends at the Post Office, traverses each street at least once, and is of shortest possible total length. This will ensure that the postman spends the minimum amount of time covering his route.

Def. Let G be a connected weighted multi-graph. A minimum postman walk of G is a closed walk which contains each edge of G and is of shortest possible total length.

$G =$ 

Ex. 2 $\langle a, b, c, d, c, a, e, c, a\rangle$ is a minimum postman walk in G of total length 17.

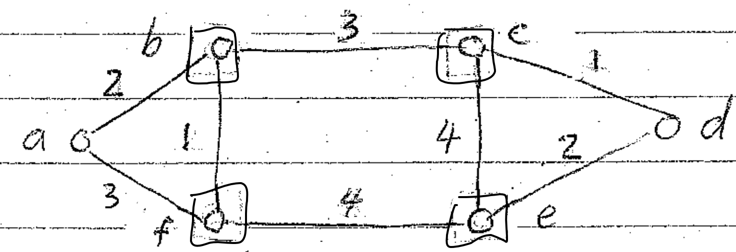
Algorithm 2 (The Postman Algorithm)

INPUT: A connected weighted multi-graph G.

OUTPUT: A minimum postman walk of G.

1. Find all the odd vertices of G and the distances between any pair of these vertices. (There is always an even no. of odd vertices in G, $2k$ say.)
2. Partition the set of odd vertices into pairs
$$\{u_1, v_1\}, \{u_2, v_2\}, \ldots, \{u_k, v_k\}$$
so that $d(u_1,v_1) + d(u_2,v_2) + \cdots + d(u_k,v_k)$ is a minimum.
3. For each i ($i = 1, \ldots, k$), add new edges of the shortest path from u_i to v_i to G to get a new multi-graph G'. Then each vertex of G' will be of even degree.
4. Find an Euler circuit Q' of G'. If we replace each of the new edges of Q' by the corresponding edge of G we will get a minimum postman walk W of G

Ex.3 Let G =



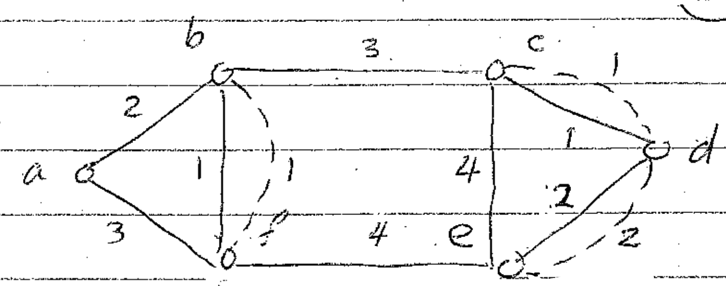
Odd vertices: b, c, e, f

$\{b,c\}$ & $\{e,f\}$ $\quad 3 + 4 = 7$

$\{b,e\}$ & $\{c,f\}$ $\quad 5 + 4 = 9$

$\{b,f\}$ & $\{c,e\}$ $\quad 1 + 3 = 4$ ✓

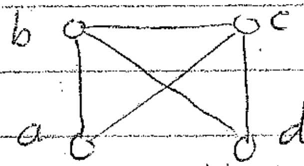
G' =



Minimum Postman walk: $a \xrightarrow{2} b \xrightarrow{1} f \xrightarrow{1} b \xrightarrow{3} c \xrightarrow{4} e \xrightarrow{2} d \xrightarrow{1} c \xrightarrow{1} d \xrightarrow{2} e \xrightarrow{4} f \xrightarrow{3} a$. Total length = 24.

§3. Hamilton cycles & ~~Hamilton~~ Hamilton paths

Def. A *Hamilton cycle* of a graph G is a cycle of G which includes each vertex of G. A *Hamilton path* of G is any path of G which includes each vertex G.

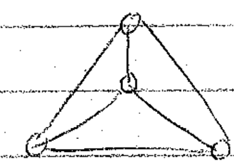
Ex. 1 Let $G =$ 

. Then

(a) $\langle a, b, c, d \rangle$ is a Hamilton path of G.

(b) $\langle a, b, d, c, a \rangle$ is a Hamilton cycle of G.

Def. A graph G is said to be *Hamilton-connected* if there is a Hamilton-path between any two distinct vertices of G.

Ex. 2 Let $G =$  . Then G is Hamilton-connected

The graph in Ex. 1 is not Hamilton-connected because there is no Hamilton path between b and c.

Theorem 4 (Dirac, 1936) If G is a graph with p vertices, $p \geq 3$, and $\deg(v) \geq p/2$ for each v in G, then G has a Hamilton cycle. (This is a special case of Prop 5.)

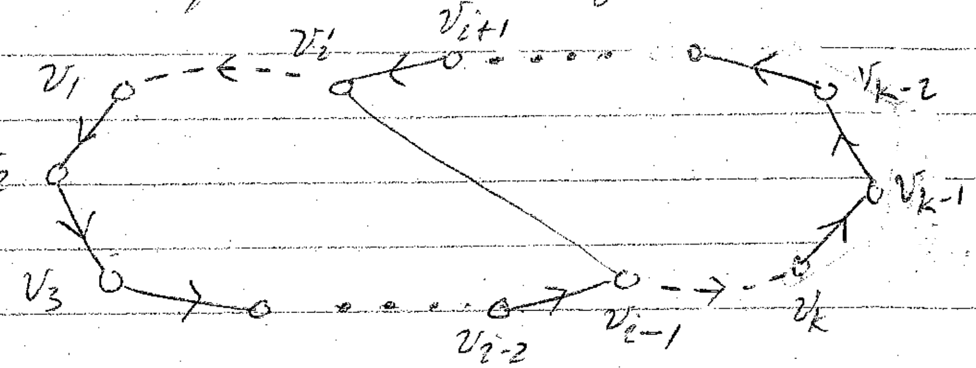
Prop. 5 (Ore's Theorem) Suppose G is a graph with p vertices, $p \geq 3$, and for each pair of non-adjacent vertices x & y,

$$\deg(x) + \deg(y) \geq p.$$

Then G has a Hamilton cycle.

Proof: Let $P_1 = \langle v_1, v_2, \ldots, v_k \rangle$ be a maximal path in G.

(i.e., a path which cannot be expanded to a longer path in G).
We will show that the vertices $\langle v_1, \ldots, v_k, v_1 \rangle$ can be rearranged, if necessary, to produce a cycle C_1.
Now, if v_k is adjacent to v_1, then $\langle v_1, v_2, \ldots, v_k, v_1 \rangle$ is the vertex sequence of our cycle C_1.



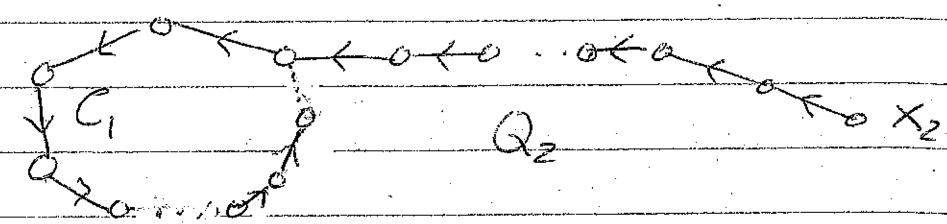
So suppose v_k is not adjacent to v_1. Then $deg(v_1) + deg(v_k) \geq p \geq k$. Since P_1 is a maximal path in G, v_1 & v_k cannot be adjacent to any vertex outside of $\{v_1, \ldots, v_k\}$. We claim that there must be an $i \in \{2, \ldots, k-1\}$ such that $v_1 v_i \in G$ and $v_{i-1} v_k \in G$. Indeed suppose this was not true. Then every time v_1 is adjacent to a vertex v_i, v_k cannot be adjacent to the vertex v_{i-1}. So $deg(v_k) \leq (k-1) - deg(v_1)$ because v_k can only be adjacent to vertices from $\{v_2, \ldots, v_k\}$ and for every degree v_1 has, v_k is denied a degree. Thus $deg(v_1) + deg(v_k) \leq k-1 \leq p-1$. But this contradicts the fact that $deg(v_1) + deg(v_k) \geq p$.

So $v_1 v_i \in G$ & $v_{i-1} v_k \in G$ for some $i \in \{2, \ldots, k-1\}$. Now it is easy to see that

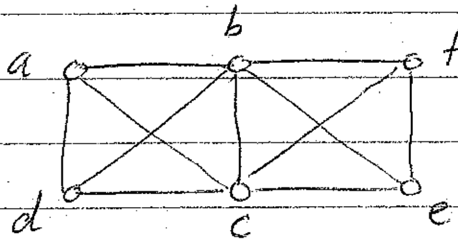
$$v_1, v_2, \ldots, v_{i-1}, v_k, v_{k-1}, \ldots, v_{i+1}, v_i, v_1$$

is a cycle C_1 in G.

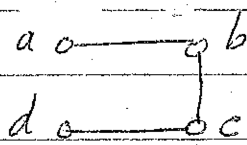
If $V(C_1) = V(G)$, then C_1 is a Hamilton cycle of G and we are done. Otherwise, choose any vertex $x_2 \in V(G) - V(C_1)$ and let Q_2 be a path containing x_2 & all of $V(C_1)$. Then extend Q_2 to a maximal path P_2 of G.

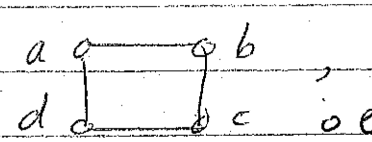


Again we can arrange to get a cycle C_2 from the maximal path P_2. If $V(C_2) = V(G)$, then C_2 will be a Hamilton cycle of G and again we will be done. Otherwise, choose a vertex $x_3 \in V(G) - V(C_2)$ and let Q_3 be a path containing x_3 & all of $V(C_2)$, and so on. In the end we will get a cycle C_n of G which is a Hamilton cycle of G because G is a finite graph.

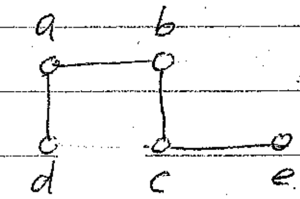
Ex. 3 Let $G =$ 

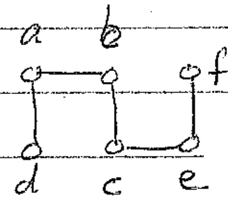
. Then

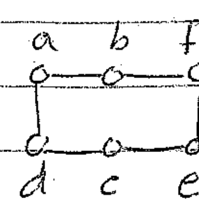
$P_1 =$ 

, $C_1 =$ 

, $x_2 = e$

$Q_2 =$ 

$P_2 =$ 

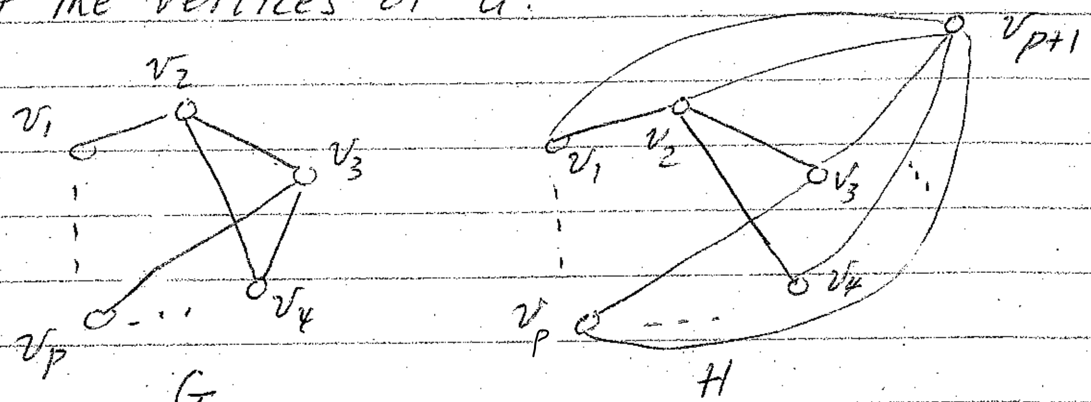
$C_2 =$ 

Corollary 6 : Let G be a graph with p vertices such that for any pair of non-adjacent vertices x and y,

$$\deg(x) + \deg(y) \geq p-1.$$

Then G has a Hamilton path.

Proof Let H be the graph obtained by adding a new vertex v_{p+1} and edges from v_{p+1} to each of the vertices of G.



Then H has $p+1$ vertices and for any pair of non-adjacent vertices x & y in H, we have

$$\deg_H(x) + \deg_H(y) = \{\deg_G(x)+1\} + \{\deg_G(y)+1\}$$
$$\geq (p-1) + 2 = p+1.$$

So by Prop. 5, H has a Hamilton cycle C of H. Now if we delete the vertex v_{p+1} from C, we will get a Hamilton path P of G.

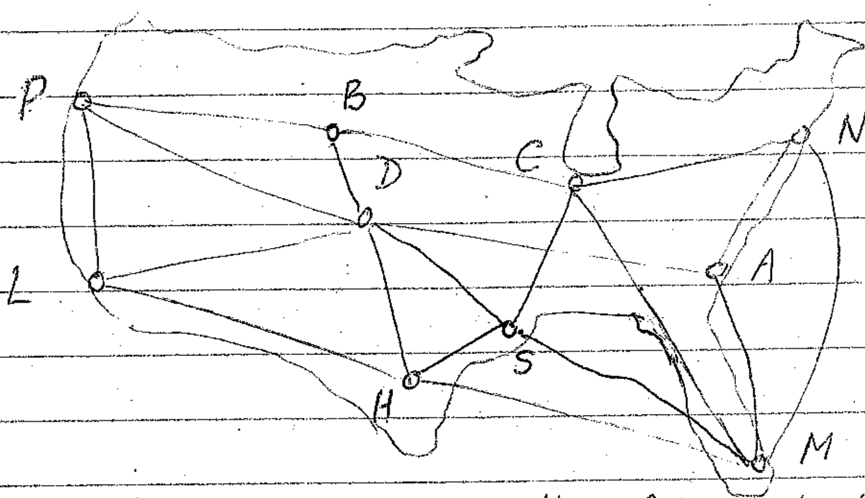
Prop. 7 Let G be a graph with p vertices such that for any pair of non-adjacent vertices x and y,

$$\deg(x) + \deg(y) \geq p+1.$$

Then G is Hamilton-connected.

Proof: Do for Homework.

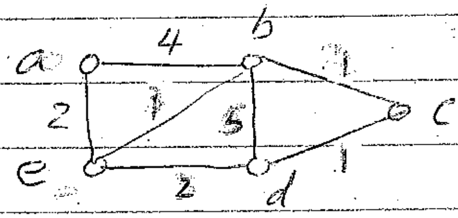
The Travelling Salesman problem.



We want a closed walk of shortest possible length which includes each city at least once.

<u>Def.</u> Let G be a weighted graph. A <u>minimum salesman walk</u> is a closed walk which includes each vertex of G and is of the shortest possible length.

<u>Ex. 4</u> Let $G =$



. Then

$a \xrightarrow{2} e \xrightarrow{1} b \xrightarrow{1} c \xrightarrow{1} d \xrightarrow{2} e \xrightarrow{2} a$ Total length $= 9$

	EDGE TRAVERSAL PROBLEMS	VERTEX TRAVERSAL PROBLEMS
1.	Euler circuit	Hamilton cycle
2.	Open Euler trail	Hamilton path
3.	Euler's Circuit theorem	No corresponding theorem
4.	Fleury's Algorithm	No corresponding algorithm
5.	Chinese Postman problem	Travelling Salesman problem.
6.	Minimum Postman walk	Minimum Salesman walk
7.	Postman Algorithm	No corresponding algorithm,