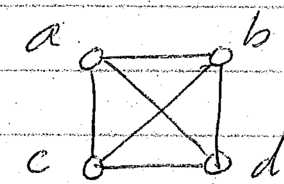
# Ch.6 – Planar graphs

## §1. Euler's Planarity formula & other properties of planar graphs

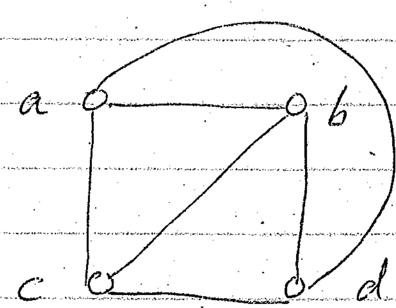
Recall that a multi-graph $G=\langle V,E\rangle$ can be represented geometrically as a subset of the plane $\mathbb{R}^2$ by using small disks for the vertices in $V$ and arcs joining two disks to represent the edges of $G$.

**Def.** A multi-graph $G$ is said to be <u>planar</u> if we can find a representation of it in the plane in which no two edges intersect. Such a representation $\mathcal{E}(G)$ is called a <u>planar embedding</u> of $G$.

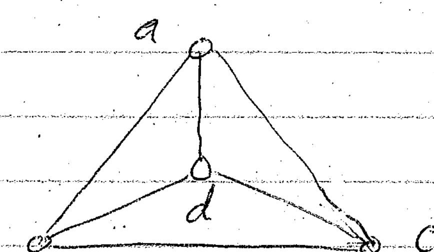
**Ex.1** Let $G = K_4 =$




. Below are two planar embeddings of $G$.



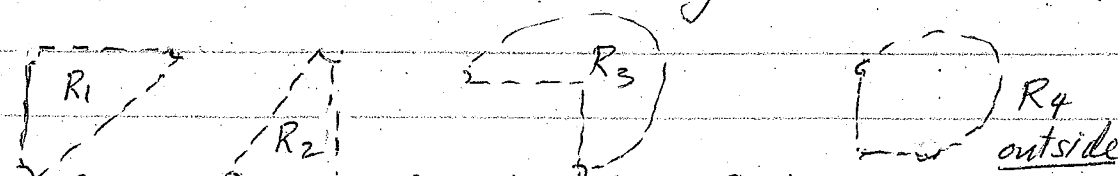

$\mathcal{E}_1(G)$




$\mathcal{E}_2(G)$

**Def.** Let $\mathcal{E}(G)$ be a planar embedding of $G$. Then $\mathbb{R}^2 - \mathcal{E}(G)$ will be a union of a finite number of <u>connected</u> <u>open</u> subsets of $\mathbb{R}^2$. Each of these open connected subsets is called a region of $\mathcal{E}(G)$.

**Ex.2**




The four regions of $\mathcal{E}_1(G)$ from Ex.1.

Def: A region is a maximal set of points S such that you can draw a continuous arc between any two points of S without ever intersecting an edge or a vertex of G.

Although the size and orientation of the regions depend on the embedding $\mathcal{E}(G)$, the number of regions depends only on G and not on the particular embedding, as we will shortly see.

Theorem 1 (Euler's planarity formula)
Let $r_{\mathcal{E}}(G)$ = the number of regions into which the planar embedding $\mathcal{E}(G)$ partitions $\mathbb{R}^2$. If G is connected, then $r_{\mathcal{E}}(G) = |E(G)| + 2 - |V(G)|$, for any $\mathcal{E}$.

Proof: We will prove the result by parametric induction on $q = |E(G)|$. First fix $p = V(G)$. Since G is connected, $q \geq p-1$.

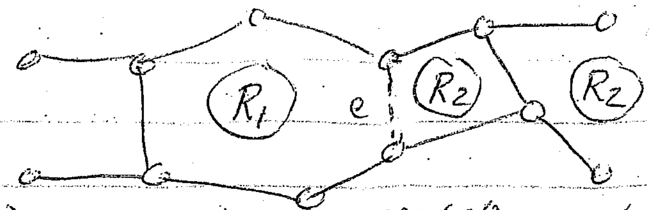
Basis: If $q = p-1$, then G is a connected multi-graph with $p-1$ edges. So G must be a tree and hence $r_{\mathcal{E}}(G) = 1$, for any planar embedding $\mathcal{E}(G)$. Since

$$1 = (p-1) + 2 - p$$

it follows that $r_{\mathcal{E}}(G) = |E(G)| + 2 - |V(G)|$. So the result is true for $q = p-1$ and for any $\mathcal{E}$.

Ind. Step. Suppose the result is true for all multi-graphs with q edges (where $q \geq p-1$) and for any $\mathcal{E}$. Let G be a connected graph with $q+1$ edges and $\mathcal{E}(G)$ be any planar embedding of G. Since G has $q+1$ edges & $q \geq p-1$, G cannot be a tree. So G must have at least one cycle, C say. Let e be any edge in C and put $G' = G - \{e\}$. Since the removal of e will reduce the number of regions

$\mathcal{E}(G)$:




of $\mathcal{E}(G)$, we have $r_{\mathcal{E}}(G') = r_{\mathcal{E}}(G) - 1$ & $|E(G')| = |E(G)| - 1$. Also $V(G) = V(G')$ & $r_{\mathcal{E}}(G') = |E(G')| + 2 - V(G')$ by the induction hypothesis. So

$$\begin{aligned} r_{\mathcal{E}}(G) &= r_{\mathcal{E}}(G') + 1 \\ &= \{|E(G')| + 2 - |V(G')|\} + 1 \\ &= \{|E(G)| - 1\} + 2 - \{|V(G)|\} + 1 \\ &= |E(G)| + 2 - |V(G)|. \end{aligned}$$

So, if the result is true for $q$, it will be true for $q+1$. Hence by the Principle of Mathematical Induction, the result is true for all $q$. Since $p$ was arbitrary, it is also true for all $p$. Hence the result is true for all planar multi-graphs.

Notation: Since $r_{\mathcal{E}}(G)$ does not depend the particular embedding $\mathcal{E}(G)$ that we use, we will denote it by just $r(G)$. We will also use $q(G)$ for $|E(G)|$ and $p(G)$ for $|V(G)|$.

Corollary 2 (Euler's Generalized Planarity formula)
Let $G$ be any planar graph & $k$ = number of connected components of $G$. Then

$$r(G) = q(G) + (k+1) - p(G).$$

Proof: Let $G_1, \dots, G_k$ be the $k$ connected components of $G$ and $\mathcal{E}(G)$ be any planar embedding of $G$. Then for each $i = 1, \dots, k$ $\quad r(G_i) = q(G_i) + 2 - p(G_i)$.

So $\sum_{i=1}^{k} r(G_i) = \sum_{i=1}^{k} q(G_i) + \sum_{i=1}^{k} 2 - \sum_{i=1}^{k} p(G_i)$.

But the infinite region is counted $k$ times (instead of just once) in the sum $\sum_{i=1}^{k} r(G_i)$. So

$$r(G) + (k-1) = q(G) + 2k - p(G)$$

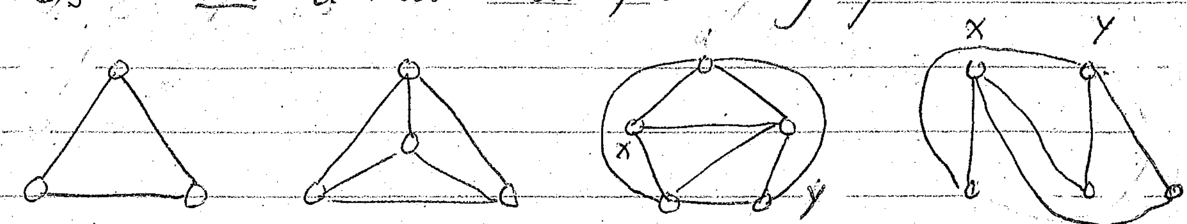
Hence $r(G) = q(G) + (k+1) - p(G)$.

<u>Def.</u> A <u>maximal planar</u> graph is any planar graph $G$ such that $G \cup \{xy\}$ is non-planar for any pair of non-adjacent vertices $x$ & $y$ in $G$.

<u>Ex. 3</u> (a) $K_3$ & $K_4$ are maximal planar graphs.

(b) $K_5 - \{\text{any edge}\}$ is a maximal planar graph

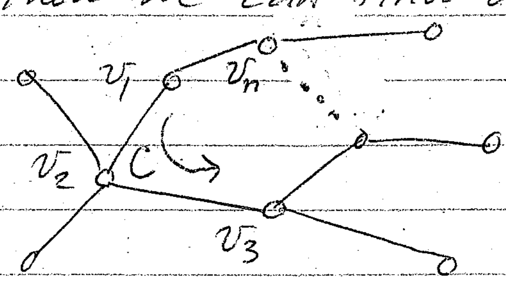
(c) $K_{2,3}$ is <u>not</u> a maximal planar graph.




$K_3$ $\quad$ $K_4$ $\quad$ $K_5 - \{xy\}$ $\quad$ $K_{2,3}$

<u>Prop. 3</u> Let $G$ be a maximal planar graph with $p \geq 3$ vertices and $\mathcal{E}(G)$ be any planar embedding of $G$. Then each region of $\mathcal{E}(G)$ is bounded by 3 edges.

<u>Proof</u>: Suppose $\mathcal{E}(G)$ has a region which is bounded by $\geq 4$ edges. Then we can find a region




which is bounded by a cycle $C = \langle v_1, v_2, v_3, \ldots, v_n, v_1 \rangle$.

There are two cases:

Case (i) $v_1 v_3 \in E(G)$. In this case the embedding of the edge $v_1 v_3$ must be outside the cycle $C$. But this means that $v_n v_2$ is prevented from being an edge in $\mathcal{E}(G)$. So we can embedd the edge $v_n v_2$ inside the cycle $C$ and hence contradict the fact that $G$ is maximal planar.

Case (ii) $v_1 v_3 \notin E(G)$. In this case $v_1$ & $v_3$ are non-adjacent vertices and we can embedd $v_1 v_3$ inside the cycle $C$ — thereby contradicting the fact that $G$ is maximal planar again.

Hence every region of $\mathcal{E}(G)$ is bounded by 3 edges.

Prop. 4 Let $G$ be any graph with $p \geq 3$ & $q = |E(G)|$.

(a) If $G$ is maximal planar, then $q = 3p-6$.

(b) If $G$ is planar then $q \leq 3p-6$.

Proof (a) Suppose $G$ is maximal planar. Let $\mathcal{E}(G)$ be any planar embedding of $G$ & $r = r(G)$. Let $A_1, \ldots, A_r$ be the regions of $\mathcal{E}(G)$. Since each region of $A_i$ is bounded by 3 edges,

$3r = e(A_1) + \cdots + e(A_r) =$ number of edges counted $= 2q$, because each edge was counted exactly 2 times.

So $3r = 2q$. But $r = q+2-p$ by Euler's planarity formula. Hence $3(q+2-p) = 2q$.

$\therefore 3q + 6 - 3p = 2q \Rightarrow q = 3p-6$.

(b) Let $G$ be a planar graph. If we add edges one at a time to $G$, we will get a maximal planar graph $G'$. So

$q = q(G) \leq q(G') = 3p(G') - 6 = 3p(G) - 6$. $\therefore q \leq 3p-6$.

## §2. Non-planar graphs & Kuratowski's theorem.

<u>Corollary 5</u> $K_5$ is a non-planar graph.

<u>Proof</u>: Suppose $K_5$ was planar. Then by Prop 4(b) we will get $q(K_5) \leq 3p(K_5) - 6$. Since $K_5$ has 10 edges and 5 vertices, this means that $10 \leq 3(5) - 6$. So $10 \leq 9$ which is a contradiction. Hence $K_5$ is non-planar.

<u>Prop. 6</u>: If $G$ is a connected planar bi-partite graph, then $q(G) \leq 2p(G) - 4$.

<u>Proof</u>: Let $\mathcal{E}(G)$ be a planar embedding of $G$ and $r = r(G)$. Since $G$ is a bipartite graph, each cycle of $G$ must have an even number of edges. Since $G$ is a graph, we need at least 3 edges to form a cycle. So each region of $\mathcal{E}(G)$ will be bounded by a cycle with at 4 edges. Let $A_1, \ldots, A_r$ be the regions of $\mathcal{E}(G)$. Then $e(A_i) \geq 4$ for each $i$.

So $4r = 4 + 4 + \cdots + 4$ ($r$ times)

$$\leq e(A_1) + e(A_2) + \ldots + e(A_r)$$

$$= \text{number of edges counted} = 2q.$$

So $2r \leq q$. But $r = q + 2 - p$. Hence

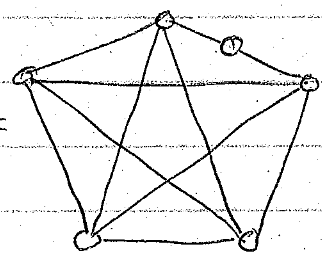
$2(q+2-p) \leq q \Rightarrow 2q + 4 - 2p \leq q \Rightarrow q \leq 2p - 4$.

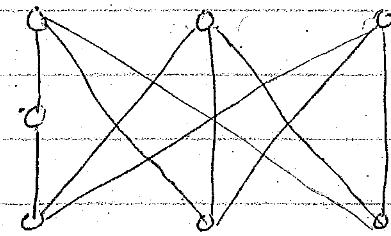
<u>Corollary 7</u>: $K_{3,3}$ is a non-planar graph.

<u>Proof</u>: Suppose $K_{3,3}$ was planar. Then by Prop 6, $q(K_{3,3}) \leq 2p(K_{3,3}) - 4$. So $9 \leq 2(6) - 4$, i.e., $9 \leq 8$ which is a contradiction. Hence $K_{3,3}$ is non-planar.

**Qu.1** When exactly is a graph non-planar?

**Ans.** We know that if $q(G) > 3p(G) - 6$, then $G$ is non-planar. Also if $G$ is bipartite & $q(G) > 2p(G) - 4$, then $G$ is also non-planar. But if $q(G) \leq 3p(G) - 6$, it does not follow that $G$ is planar. Also if $G$ is bipartite & $q(G) \leq 2p(G) - 4$, it does not follow that $G$ is planar.

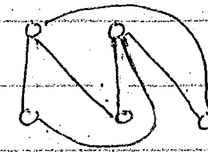
**Ex. 1** (a) $G_1 =$ 

(b) $G_2 =$ 

$q(G_1) = 11 \leq 3(6) - 6 = 3p(G_1) - 6$ $\quad$ $q(G_2) = 10 \leq 2(7) - 4 = 2p(G_2) - 4$.

It is easy to see that if $G_1$ & $G_2$ were planar then $K_5$ & $K_{3,3}$ will be also be planar. So $G_1$ & $G_2$ are non-planar.

Since $K_5$ & $K_{3,3}$ are non-planar, any graph $G$ that contains $K_5$ or $K_{3,3}$ as a subgraph (or something that "amounts" to being a subgraph) will be non-planar. So $K_6, K_7, K_8, \ldots$ and $K_{3,4}, K_{3,5}, \ldots, K_{4,4}, K_{4,5}, \ldots, K_{5,5}$ are all non-planar.

**Qu.2** (a) Is $K_{2,3}$ planar? Yes. → 

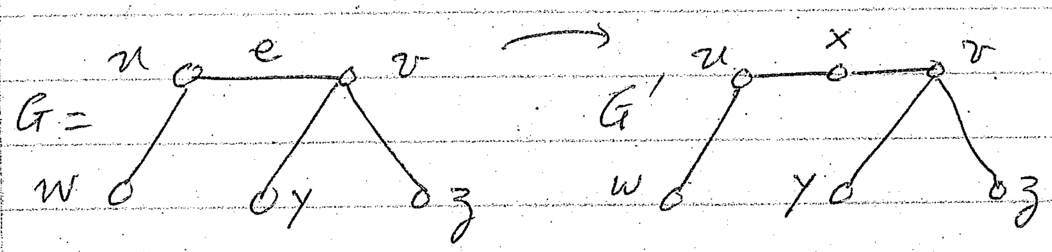
(b) Is $K_{2,2,2}$ planar? Yes. Do for H.W.

(c) Is $K_{2,2,3}$ planar? No. $16 \not\leq 3(7) - 6$

(d) Is $K_{2,2,2,2}$ planar? No. $24 \not\leq 3(8) - 6$

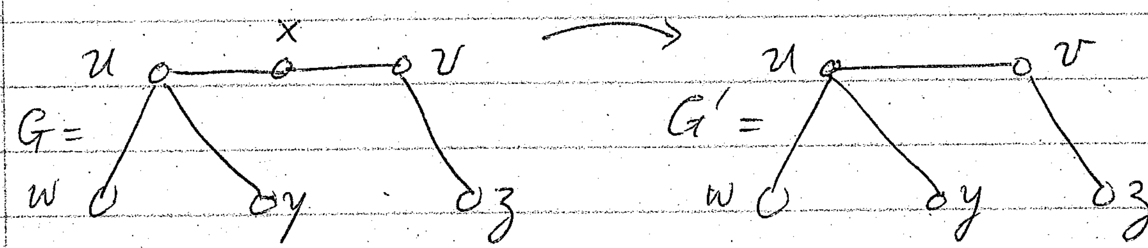
**Def.** Let $e = uv$ be an edge in a graph $G$. Then we can create a vertex of degree 2 on the edge $e$ by adding a new vertex $x$ to $G$, by adding the edges $ux$ & $xv$, and by deleting the edge $uv$ from $G$.

**Ex.2**




**Def.** Let $x$ be a vertex of degree 2 in a graph $G$. Then we can merge out the vertex $x$ from $G$ by deleting the vertex $x$ and by adding a new edge between the two vertices that were adjacent to $x$ in $G$.

**Ex.3**




**Def.** Two graphs $G$ & $H$ are homeomorphic if we can transform $G$ into $H$ by creating vertices of degree 2 on certain edges of $G$ or by merging out certain vertices of degree 2 in $G$.

Theorem 8 (Kurotowski's planarity theorem). $G$ is planar $\Leftrightarrow$ $G$ has no subgraph which is homeomorphic to $K_5$ or $K_{3,3}$.

Proof: ($\Rightarrow$) Suppose $G$ is planar. Then $G$ cannot contain any subgraph which is homeomorphic to $K_5$ or $K_{3,3}$ (otherwise $K_5$ or $K_{3,3}$ would be planar. ($\Leftarrow$): hard - see textbook.

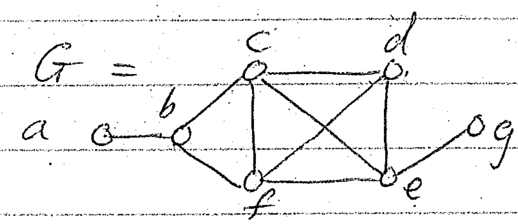
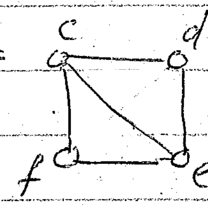
## §3. The Demoucron, Malgrange & Pertuiset Planarity Algorithm

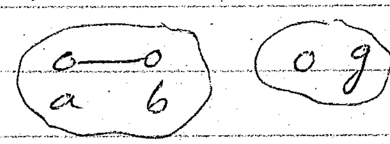
**Def.** Let $G$ be a graph and $H$ be a subgraph of $G$. A <u>piece</u> of $G$ relative to $H$ is either

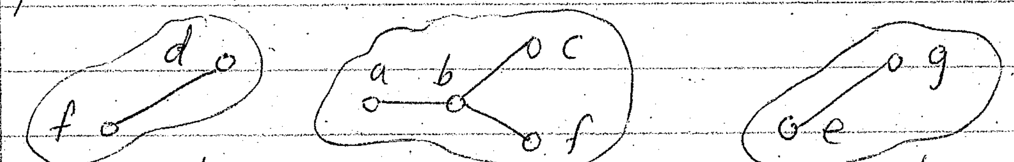
(a) an edge $e = uv$ with $e \notin E(H)$ & $u, v \in V(H)$, <u>or</u>

(b) a component $C$ of $G - V(H)$ plus all the edges joining vertices of $C$ to vertices of $H$.

**Def.** Let $P$ be a piece of $G$ relative to $H$. If $v \in V(P) \cap V(H)$, we say that $v$ is a <u>contact vertex</u> of $P$. If the piece $P$ has 2 or more contact vertices, we say that $P$ is a <u>segment</u> of $G$ relative to $H$.

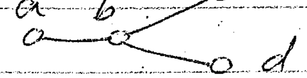
**Ex. 1** Let $G =$  and $H =$ 

Then $G - V(H) =$  . So the pieces of $G$ relative are as shown below.



segment     segment     not a segment.

**Def.** Recall that a <u>cut-vertex</u> of $G$ is any vertex $v$ of a connected graph such that $G - \{v\}$ is disconnected. A connected graph with no cut-vertex is called a <u>block</u> (or <u>connected block</u>).

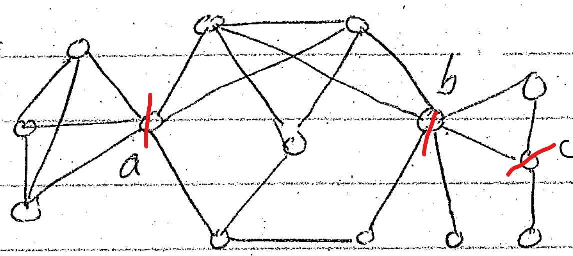
**Ex. 2** $G =$  blocks of $G$: 

The DMP Planarity algorithm will take a connected block as input. So before we apply the algorithm we must first pre-process the graph, we are testing for planarity.

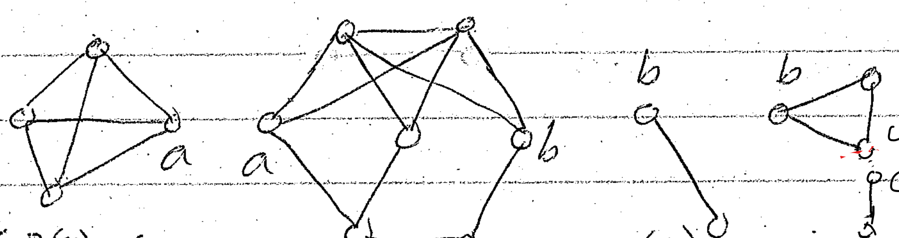
## Pre-processing G for the DMP Planarity Algorithm

1. If G is not connected, then consider each component separately.
2. If a connected component has cut-vertices, then split the cut-vertices to get a set of blocks of G
3. If $q(G_i) > 3p(G_i) - 6$ for any block $G_i$ with $p(G_i) \geq 3$, then that block is non-planar & so G is non-planar.

Ex. 3 Let G =




. Then the 5 blocks of G are shown below.




$3 \leq 3(3) - 6$

$6 \leq 3(4) - 6$ $11 \leq 3(7) - 6$ $p(G_i) < 3$. $p(G_i) < 3$

## Algorithm 1 (The DMP Planarity algorithm)

INPUT : A pre-processed block $G = \langle V, E \rangle$

OUTPUT : $\begin{cases} \text{A planar embedding of } G, & \text{if } G \text{ is planar} \\ \text{NON-PLANAR}, & \text{if } G \text{ is non-planar.} \end{cases}$

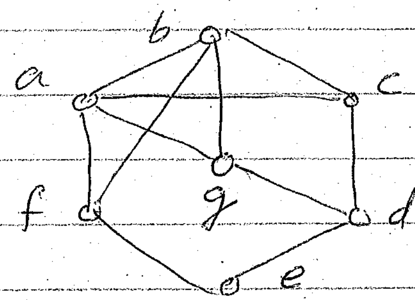
1. If G has no cycles, then G must be the tree $K_2$ and o—o is a planar embedding of G & we are done. Otherwise, choose any cycle C in G. Let $i \leftarrow 1$, $r \leftarrow 2$, and $H_i \leftarrow$ a planar embedding of C.

2. If $E(H_i) = E(G)$, STOP. Otherwise, find all the segments of $G$ relative to $H_i$ and for each segment $S$, let $\mathcal{R}_i(S) \leftarrow$ the set of regions of $H_i$ into which $S$ can be compatibly embedded.

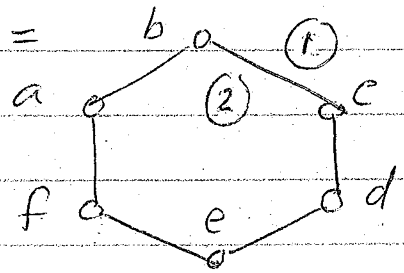
3. If there is a segment $S$ such that $\mathcal{R}_i(S) = \emptyset$, then say NON-PLANAR and STOP;
If there is a segment $S$ such that $|\mathcal{R}_i(S)| = 1$, then let $R \leftarrow$ the unique region in $\mathcal{R}_i(S)$;
Otherwise, choose any segment $S$ and let $R \leftarrow$ any one of the regions in $\mathcal{R}_i(S)$.

4. Choose any path $L$ in $S$ which connects two contact vertices of $S$. Then let
$$H_{i+1} \leftarrow H_i \cup \{\text{the embedding of } L \text{ in the region } R\},$$
$i \leftarrow i+1$, $r \leftarrow r+1$, and go to step 2.

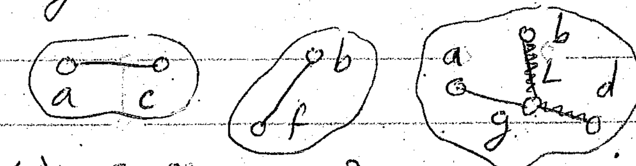
Ex.3 Determine whether or not the graph on the right is planar?




Sol $H_1 =$

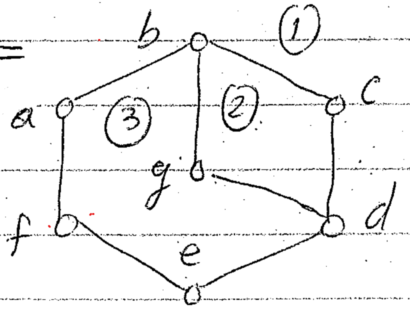


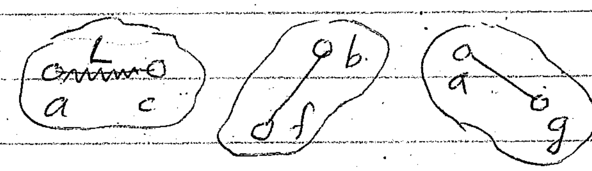

Segments of $G$ relative to $H_i$




$\mathcal{R}_1(S)$: $\{1,2\}$ $\{1,2\}$ $\{1,2\}$
↑
R

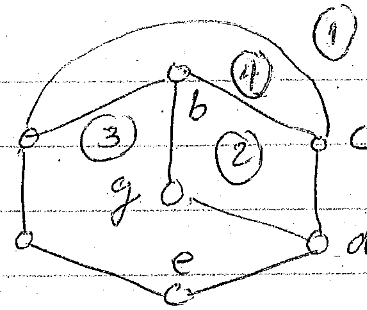
$H_2 =$

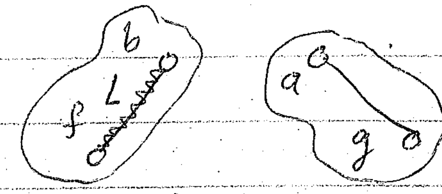



$\mathcal{R}_2(S)$: $\{1\}$ $\{1,3\}$ $\{3\}$
↑
R

$H_3 =$

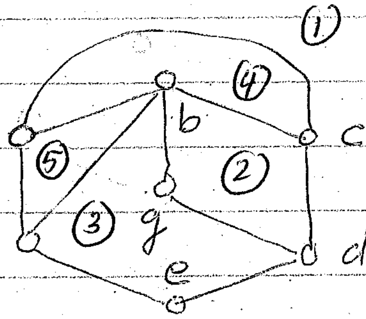


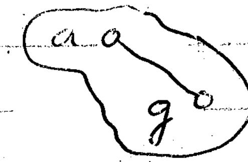




$R_3(S)$: $\{3\}$ $\{3\}$

↑
R

$H_4 =$

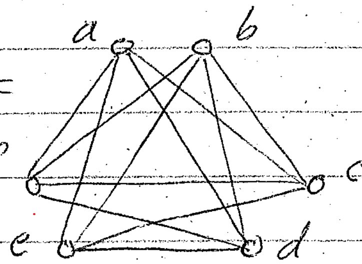


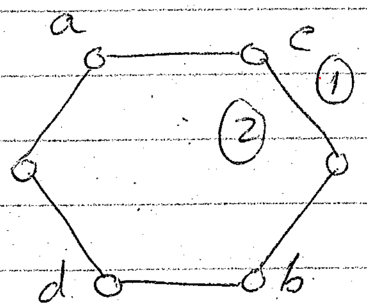

$R_4(S)$: $\emptyset$ STOP. NON-PLANAR

Ex.4 Determine whether or not $K_{2,2,2}$ is planar.

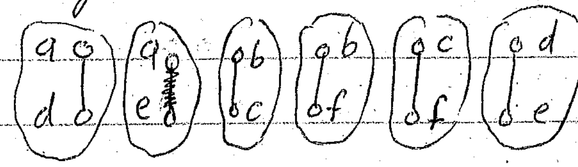
$G =$




Sol. $H_1 =$



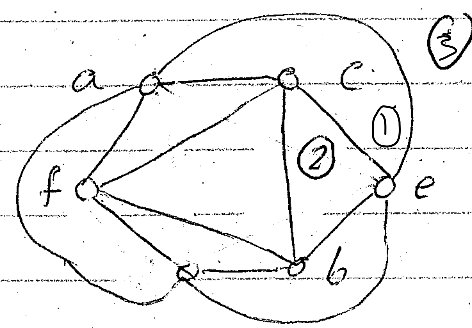

Segments of $G$ rel. to $H_1$




↑
R

⋮

$H_7 =$

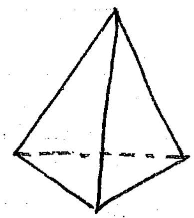



So $K_{2,2,2}$ is planar.

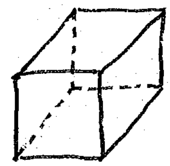
# §4. Polyhedral graphs & the geometric dual

A *polyhedron* is a solid figure with plane polygonal faces that can be continuously distorted (transformed) into a solid sphere.

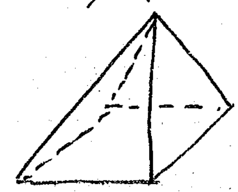
**Ex.1** Some polyhedra (some textbooks call these simple polyhedra)



tetrahedron

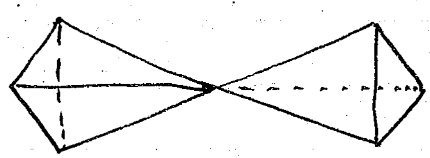


cube

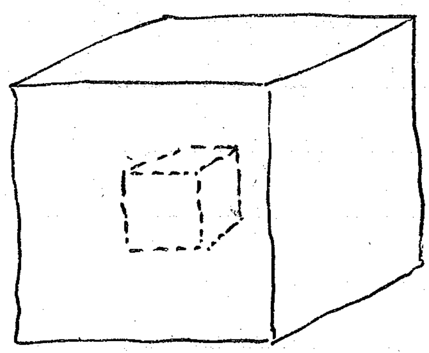


square based pyramid

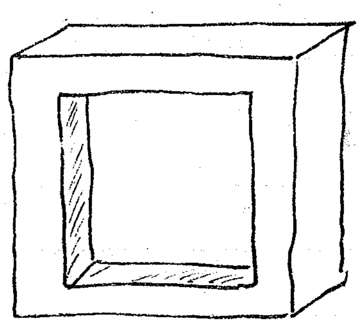
**Ex.2** Some solids that are not polyhedra



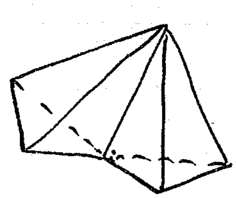
two tetrahedra joined at a vertex.



Large cube with a smaller cube hollowed out in the center.



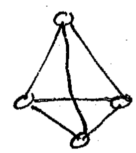
The picture frame. (Solid cube with a hole drilled through front to back face)



Two tetrahedra welded together along an edge.

A <u>polyhedral graph</u> is any graph that can be obtained by considering the vertices and edges of a polyhedron as the vertices and edges of a graph.

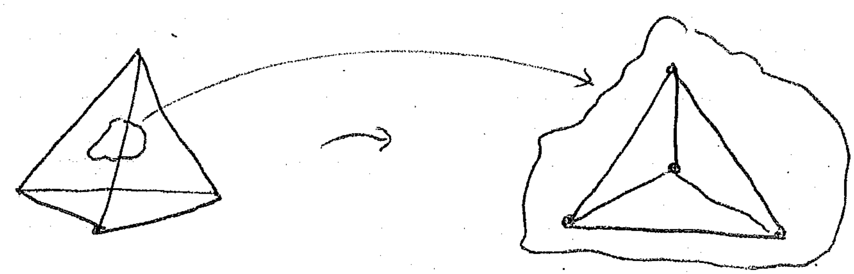
<u>Ex. 3</u>



tetrahedral graph.

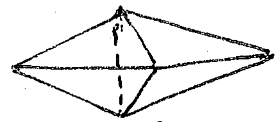
<u>Prop 8</u> If $G$ is a polyhedral graph, then $G$ is planar and obviously connected

If $G$ is a polyhedral graph, consider the polyhedron from which $G$ was obtained. If we imagine that the polyhedron is hollow and we make a hole in one face and stretch the polyhedron onto the plane, we will get a planar embedding of $G$.



<u>Def.</u> A <u>regular polyhedron</u> is one in which each face is a <u>fixed</u> regular polygon and in which each vertex has the same no. of edges incidented to it.

<u>Ex 4</u>. The tetrahedron and cube are regular.

Qu: Is  a regular polyhedron?
(assume each face is an equilateral triangle) — (NO)

Is it even a polyhedron? (YES).

Theorem 9: There are exactly 5 regular polyhedra.

Proof: Suppose $P$ is a regular polyhedron. Then each face of $P$ is a fixed regular polygon with $k$ edges, say. Let $A_1, \ldots, A_r$ be the faces of $P$. Then

$$e(A_1) + \cdots + e(A_r) = 2q.$$

So $$k \cdot r = 2q \quad \text{-----} \quad (1)$$

Also each vertex has a fixed number, $\ell$ say, of edges incidented to it. So the degree of each vertex is $\ell$. Now by the first Theorem of graph Th., sum of degrees $= 2$ (no. of edges)

So $$\ell \cdot p = 2q \quad \text{....} \quad (2)$$

Also by Euler's formula $r = q + 2 - p$ ... (3)

Now from (1) we have $r = 2q/k$, and from (2) we have $p = 2q/\ell$. Substituting in (3) we get

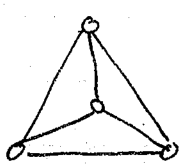
$$\frac{2q}{k} = q + 2 - \frac{2q}{\ell}$$

So $$\frac{2q}{k} + \frac{2q}{\ell} = \frac{2q}{2} + \frac{2q}{q}$$

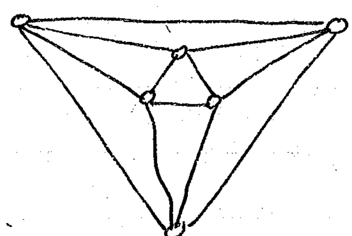
Hence $$\frac{1}{k} + \frac{1}{\ell} = \frac{1}{2} + \frac{1}{q} \quad \text{......} \quad (4)$$

Now we know that $k \geq 3$ (a polygon can't have less than 3 edges) and $l \geq 3$ (for the figure to be solid we need at least 3 edges at each vertex, and from (4) we get $\frac{1}{k} + \frac{1}{l} > \frac{1}{2}$. So the only possible values of $k$ and $l$ are given in the table below.

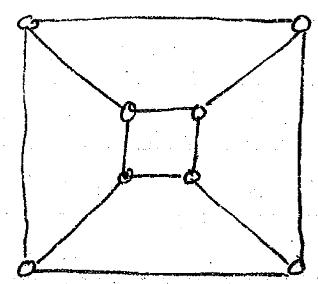
| $k$ | $l$ | $q$ | $p$ | $r$ | | | |
|---|---|---|---|---|---|---|---|
| 3 | 3 | 6 | 4 | 4 | tetra- | hedron | |
| 3 | 4 | 12 | 6 | 8 | octa- | " | |
| 3 | 5 | 30 | 12 | 20 | icosa- | " | |
| 4 | 3 | 12 | 8 | 6 | hexa- | " | = cube |
| 5 | 3 | 30 | 20 | 12 | dodeca- | " | |



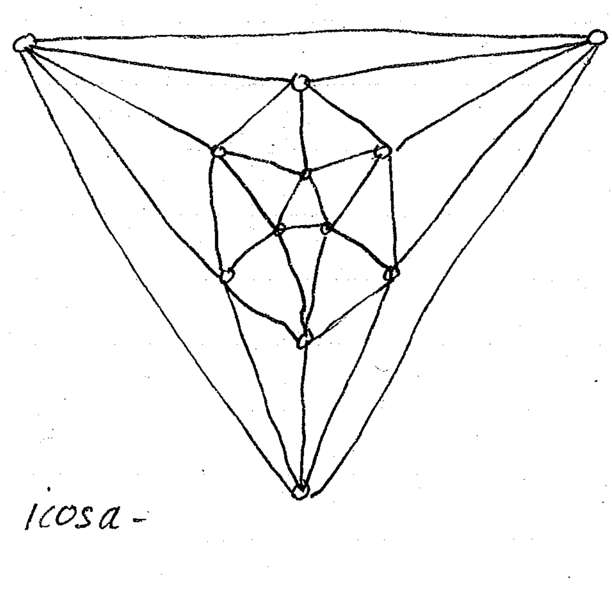
tetra-



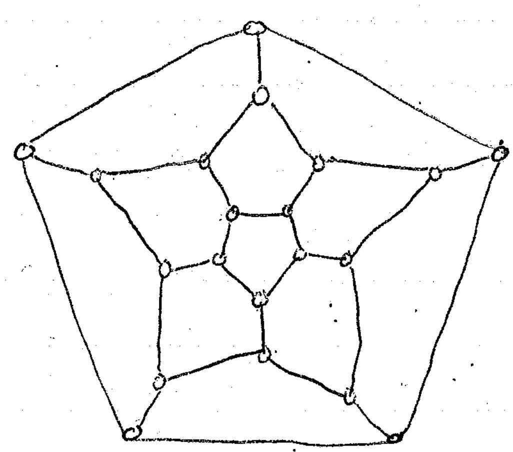
octa-



hexa-



icosa-



dodeca-

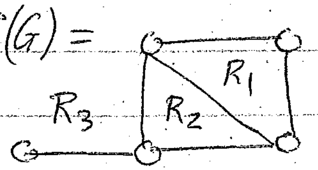
## §5. The geometric dual & graphs on other surfaces

**Def.** Let $\mathcal{E}(G)$ be a planar embedding of a planar graph $G$. We define the <u>geometric dual</u> $G^*_{\mathcal{E}}$ of $G$ by

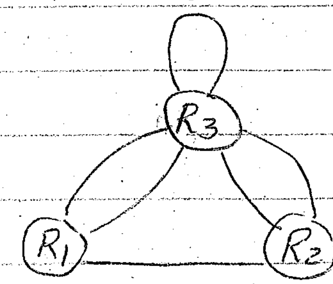
(a) $V(G^*_{\mathcal{E}})$ = set of the regions into which $\mathcal{E}(G)$ partitions the plane $\mathbb{R}^2$.

(b) For each edge $e$ in $\mathcal{E}(G)$ that is a common boundary of the regions $R_1$ & $R_2$, we get an edge between $R_1$ & $R_2$ in $G^*_{\mathcal{E}}$.

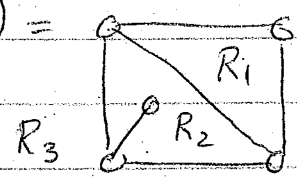
**Ex. 1 (a)** Let $\mathcal{E}(G)$ =



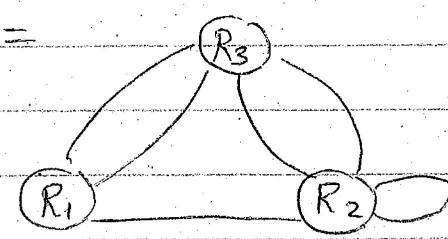

. Then $G^*_{\mathcal{E}}$ =




**(b)** Let $\mathcal{E}'(G)$ =




. Then $G^*_{\mathcal{E}'}$ =




<u>Note</u>

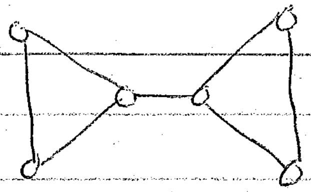
From Ex. 1, it is clearly that $G^*_{\mathcal{E}}$ depends on the particular embedding of $G$ that is selected. In general, $G^*_{\mathcal{E}}$ will be a multi-pseudo-graph.

**Qu. 1** When is $G^*_{\mathcal{E}}$ independent of the embedding $\mathcal{E}$?

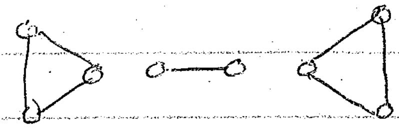
**2.** When is $G^*_{\mathcal{E}}$ guaranteed to be a graph?

**Def.** The graph $G$ is <u>1-isomorphic</u> to $H$ if we can split $G$ into blocks (by splitting its cut-vertices) and refit the blocks (by identifying pairs of vertices) to get $H$.

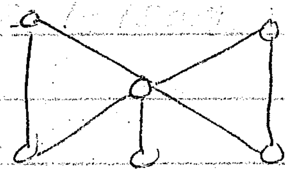
Ex. 2 Let $G =$



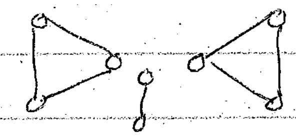
. Blocks of $G$:



& $H =$



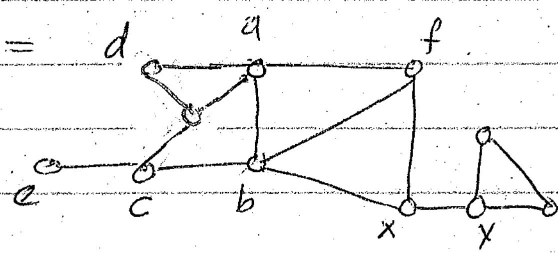
Blocks of $H$:



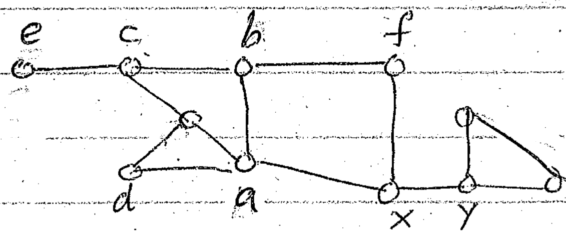
Then $G$ is 1-isomorphic to $H$.

<u>Def.</u> The graph $G$ is <u>2-isomorphic</u> to $H$ if by flipping around a portion of $G$ (which can be separated by splitting two vertices) we can get a graph $G'$ which is 1-isomorphic to $H$.

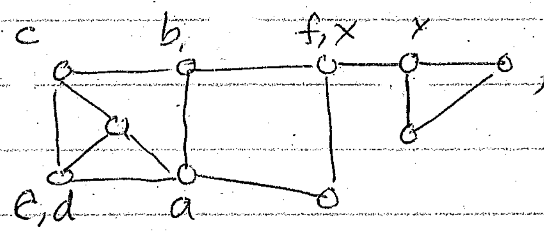
<u>Ex 3</u> Let $G =$




Then $G' =$




& if $H =$




, then $G$ is 2-isomorphic to $H$.

<u>Theorem 10</u>: If $G$ is a planar graph with $k_v(G) \geq 3$, then

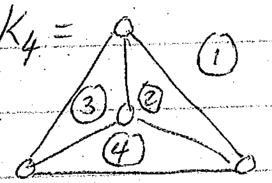
(a) $G^*_\varepsilon$ will be independent of the embedding $\varepsilon$

(b) $G^*_\varepsilon$ will always be a graph,

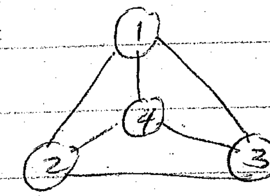
(c) $(G^*)^* \cong G$.

<u>Def.</u> Let $G$ be a graph with $k_V(G) \geq 3$. Then $G^*_{\mathcal{E}}$ does not depend on $\mathcal{E}$. So we will denote $G^*_{\mathcal{E}}$ by $G^*$. We say that $G^*$ is self-dual if $G^* \cong G$.

<u>Ex.4</u> Let $G = K_4 =$

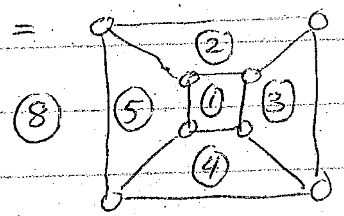



. Then $G^* =$

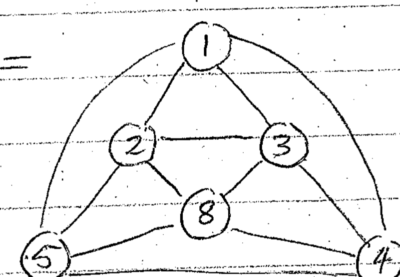



So $K_4$ is self-dual.

<u>Ex.5</u> Let $G =$




. Then $G^* =$




and $(G^*)^* \cong G$. So the geometrical dual of the cube graph is the octahedral graph & the geometrical dual of the octahedral graph is the cube graph.

<u>Ex.6</u> Show that $(\text{icosahedral graph})^* =$ dodecahedral graph and $(\text{dodecahedral graph})^* =$ icosahedral graph.

<u>Graphs on other surfaces</u>

1. A graph that can be embedded (with no edges crossing) on the surface of a sphere is called a <u>spheroidal graph</u>

<u>Fact 11</u> $G$ is spheroidal $\Leftrightarrow$ $G$ is planar

2. A graph that can be embedded (with no edges crossing) on the surface of a torus is called a <u>toroidal graph</u>

<u>Fact 12</u>: $K_5$, $K_{3,3}$, $K_{3,4}$, $K_6$ & $K_7$ are all toroidal graphs.

<u>Extra H.W.</u> 1. Find the dual polyhedron to the hexahedron on p.15
2. Find two more self-dual planar graphs.