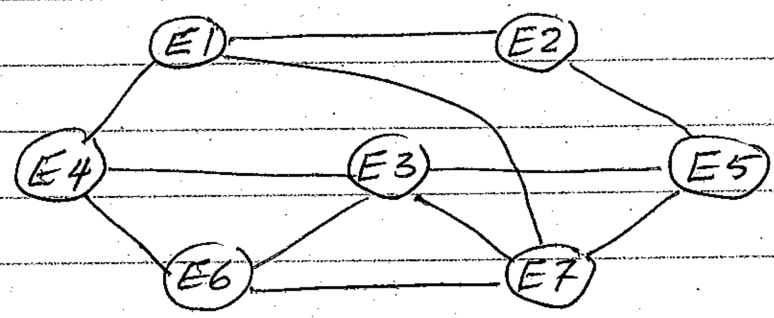
Ch.7 - Colorings & Matchings in graphs

§1. Legal colorings in graphs

At the end of each semester, students are required to take final exams in each of their courses. Each exam is given in a 2-hour period an, naturally, two exams cannot be scheduled for the same time period, if there are students who need to take both of those exams. What is the minimum no. of exam periods needed to ensure no conflicts

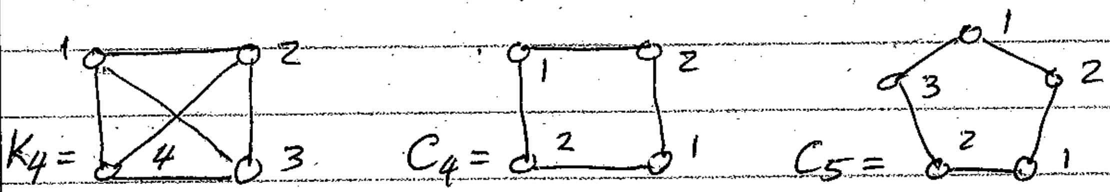


Time Period	Exams
1st 2hrs	E1, E3,
2nd "	E2, E4, E7
3rd "	E5, E6

If we designate the 1st, 2nd, 3rd, 4th ... periods by the colors RED, YELLOW, GREEN, BLUE etc, what we want is a coloring of the vertices in which no two adjacent vertices have the same color.

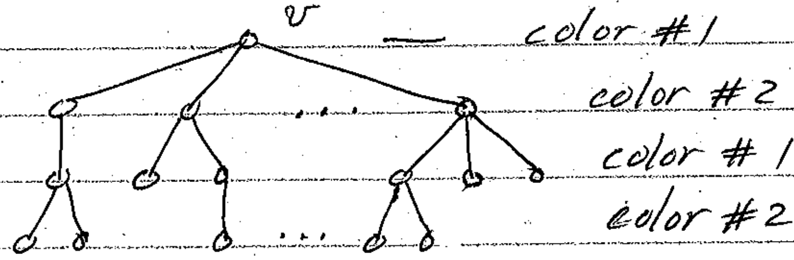
Def. A legal coloring of a graph G is any function $f: V(G) \to \mathbb{Z}^+ = \{1, 2, 3, \ldots\}$ such that $f(u) \neq f(v)$ for any two adjacent vertices u & v. The chromatic number $\chi(G)$ of G is the smallest number of colors that can be used to get a legal coloring of G. If $\chi(G) = k$, then we say that G is k-colorable.

Ex. 1 $\chi(K_4) = 4$, $\chi(C_4) = 2$, $\chi(C_5) = 3$.



Prop. 1: If T is a non-trivial tree, then $\chi(T) = 2$.

Proof: If T is a non-trivial tree, then $|E(T)| \geq 1$. Since we need at least 2 colors for the endpoints of an edge, $\chi(T) \geq 2$. Now select any vertex v of T and designate it as the root. Then $\langle T, v \rangle$ will be a rooted tree. Color the vertices in the even levels with color #1 & the vertices in the odd levels with color #2. This will give us a legal coloring of T because no two vertices in the even levels are adjacent & the same is true for the vertices in the odd levels. So $\chi(T) \leq 2$. Hence $\chi(T) = 2$.



Prop. 2 $\chi(G) \leq 2 \Leftrightarrow G$ contains no cycles of odd length.

Proof: ($\Rightarrow$) Suppose $\chi \leq 2$. Then contains no cycles of odd length (because such a cycle alone needs 3 colors)

($\Leftarrow$) Suppose G contains no cycles of odd length. Let F be a spanning forest of G. Color the vertices of each tree T_i in F according to the method used in Prop. 1.

Now add the edges of G, one at a time. This will not invalidate the coloring because each time we add an edge, it must join two vertices with different colors, otherwise we will get an odd cycle in G. So $\chi(G) \leq 2$.

Theorem 3: For any graph G, $\chi(G) \leq \Delta(G)+1$.

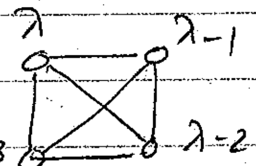
Proof: We will prove the result by induction on $p=|V(G)|$. If $p=1$, then $G \cong K_1$ and since $\chi(K_1) = 1 \leq 0+1$, it follows that the result is true for $p=1$. Now suppose the result is true for all graphs with $p \geq 1$ vertices. Let G be any graph with $p+1$ vertices. Choose any vertex v in G and let $G' = G-\{v\}$. Then $\Delta(G') \leq \Delta(G)$. Since G' has p vertices, we can find a legal coloring $f: V(G') \to \{1, \cdots, k\}$ of G' where $k = \Delta(G)+1$, because $\Delta(G') \leq \Delta(G)$. Now v will be adjacent to at most $\Delta(G)$ vertices in G, so we can always find a color among the k colors which was not used on these adjacent vertices. If assign this color to the vertex v, we will get a legal coloring of G. So $\chi(G) \leq k = \Delta(G)+1$. So if the result is true for p, it will be true for $p+1$. Hence, by the Principle of Mathematical Induction, the result is true for all graphs.

Theorem 4. (Vizing's theorem). If G is a graph which is not isomorphic to K_p or to C_{2p+1}, then $\chi(G) \leq \Delta(G)$.

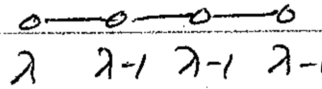
Proof: (quite hard). See p.228 of the textbook.

§2 The Chromatic Polynomial & the Four-color theorem.

Qu.1 Suppose G is a graph and we have λ available colors. In how many ways can we legally color G?

Ex1(a) Let $G = K_4 =$ 

. Then G can be legally colored in $\lambda(\lambda-1)(\lambda-2)(\lambda-3)$ different ways.

(b) Let $G = L_4 =$ 

. Then G can be legally colored in $\lambda(\lambda-1)(\lambda-1)(\lambda-1) = \lambda(\lambda-1)^3$ ways.

Def. The chromatic polynomial of a graph G is defined by $P_G(\lambda) =$ No. of ways G can be legally colored with λ available colors.

Ex2(a) $P_{N_n}(\lambda) = \lambda(\lambda)(\lambda)\cdots(\lambda)$ (n times) $= \lambda^n$

(b) $P_{K_n}(\lambda) = \lambda(\lambda-1)(\lambda-2)\ldots(\lambda-(n-1)) = \lambda^{(n)} = \dfrac{\lambda!}{(\lambda-n)!}$

Prop 5. If T is a tree with n vertices, then $P_T(\lambda) = \lambda(\lambda-1)^{n-1}$.

Proof: We will prove the result by induction on $n = |V(T)|$. If $n=1$, then $T = K_1$, so $P_T(\lambda) = \lambda = \lambda(\lambda-1)^{1-1}$. So the result is true for $n=1$.

Now suppose the result is true for all trees with n vertices. Let T be a tree with $n+1$ vertices and choose any leaf v_0 in T. Put $T' = T - \{v_0\}$. Then

$$P_T(\lambda) = (\lambda-1) . P_{T'}(\lambda) = (\lambda-1) . \lambda(\lambda-1)^{n-1} = \lambda . (\lambda-1)^{n+1-1}.$$

So if the result is true for n, it will be true for $n+1$. Hence the result is true for all trees by the Princ. of Math Ind.

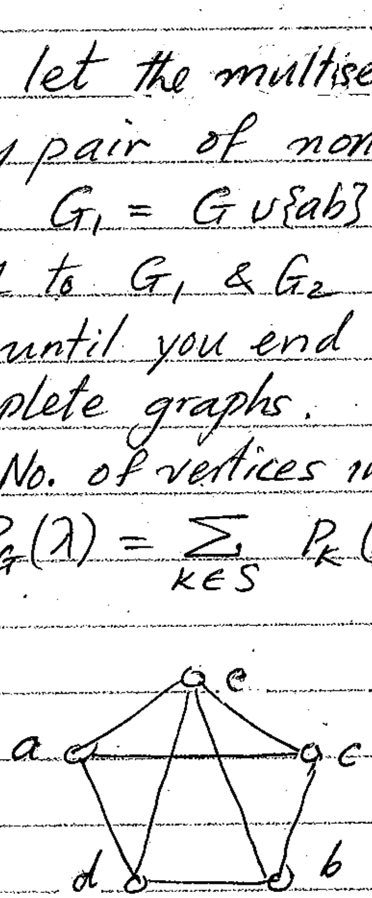
Theorem 6: Let G be a graph with two non-adjacent vertices a & b. Then $P_G(\lambda) = P_{G \cup \{ab\}}(\lambda) + P_{G \circ \{ab\}}(\lambda)$ where $G \circ \{ab\}$ is the graph obtained by considering the vertices a & b to be identical.

Proof: $P_G(\lambda)$ = No. of ways of coloring G with λ available colors
= No. of ways of coloring G with a & b colored differently
+ No. of ways of coloring G with a & b colored the same
$= P_{G \cup \{ab\}}(\lambda) + P_{G \circ \{ab\}}(\lambda)$.

Algorithm 1 (The Chromatic Polynomial algorithm)
INPUT: A graph with n vertices
OUTPUT: $P_G(\lambda)$ and $\chi(G)$.

1. If $G = K_n$, let the multiset $S = [K_n]$; otherwise choose any pair of non-adjacent vertices a & b and let $G_1 = G \cup \{ab\}$ & $G_2 = G \circ \{ab\}$.
2. Apply step 1 to G_1 & G_2 separately and continue until you end up with a multi-set S of complete graphs.
3. Let $\chi(G)$ = No. of vertices in the smallest complete graph in S. Also put $P_G(\lambda) = \sum_{K \in S} P_K(\lambda)$.

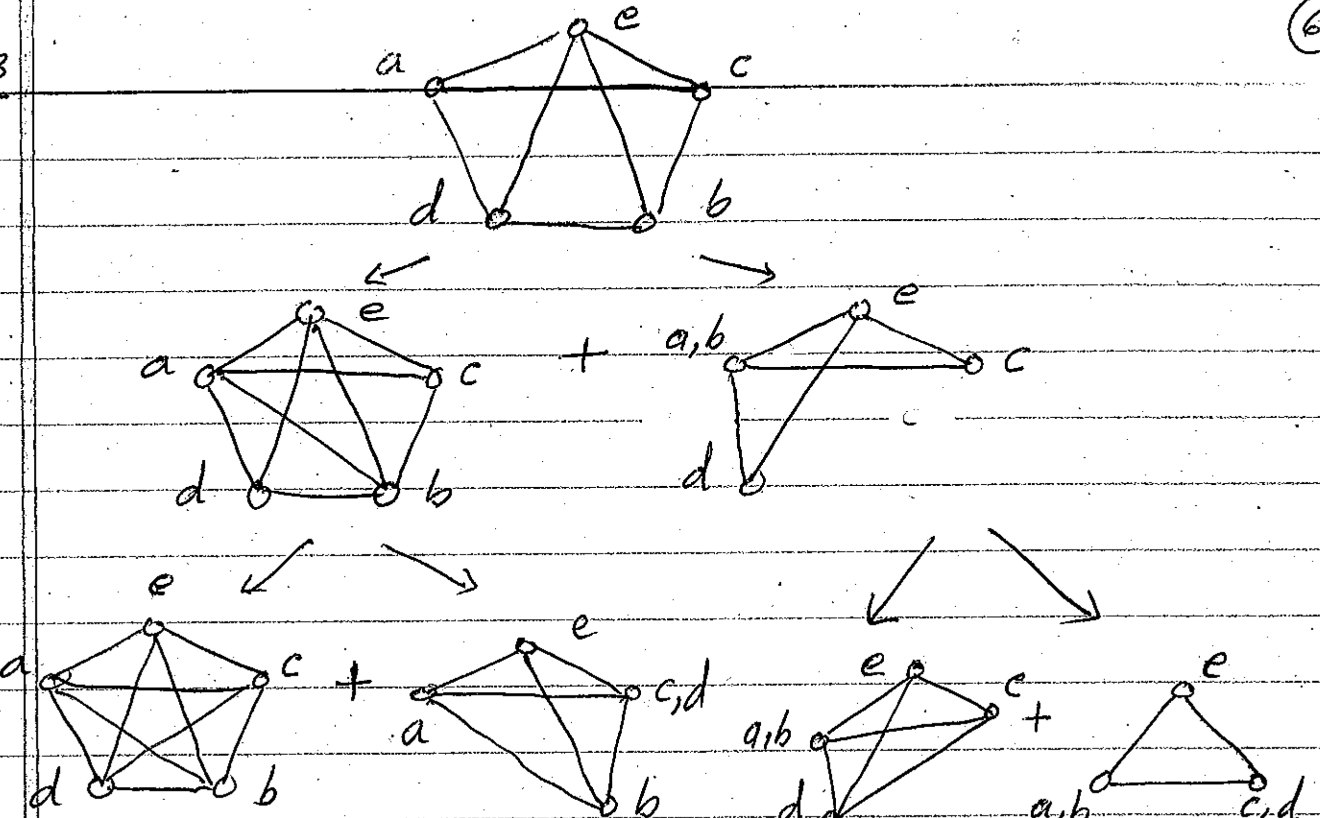
Ex3 Let $G =$



. Find (i) $P_G(\lambda)$ and (ii) $\chi(G)$.

(iii) In how many ways can G be legally colored with 6 colors available.

Ex. 3



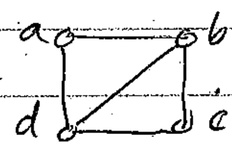
$\therefore$ $S = [K_5, 2 \cdot K_4, K_3]$. So (i) $\chi(G) = 3$.

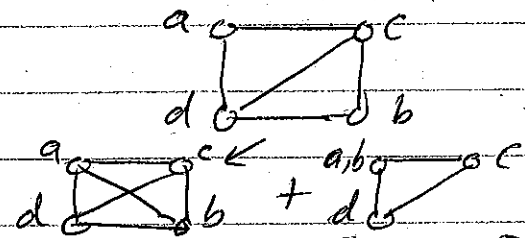
(ii) $P_G(\lambda) = P_{K_5}(\lambda) + P_{K_4}(\lambda) + P_{K_4}(\lambda) + P_{K_3}(\lambda)$

$$= \lambda^{(5)} + 2 \cdot \lambda^{(4)} + \lambda^{(3)}$$
$$= \lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4) + 2\lambda(\lambda-1)(\lambda-2)(\lambda-3) + \lambda(\lambda-1)(\lambda-2)$$
$$= \lambda(\lambda-1)(\lambda-2)[(\lambda^2-7\lambda+12) + (2\lambda-6) + 1]$$
$$= \lambda(\lambda-1)(\lambda-2)(\lambda^2-5\lambda+7).$$

(iii) Number of ways G can be legally colored with 6 colors available $= P_G(6) = 6 \cdot 5 \cdot 4 \cdot (6^2 - 5(6) + 7)$

$$= (120)(13) = 1{,}560.$$

Ex. 4 Let $G =$ 



Find $P_G(\lambda)$ & $\chi(G)$.

So $\chi(G) = 3$ and $S = [K_4, K_3]$

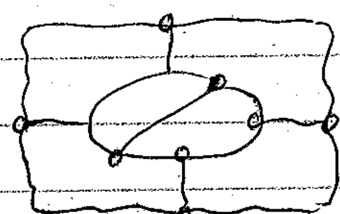
$P_G(\lambda) = P_{K_4}(\lambda) + P_{K_3}(\lambda) = \lambda(\lambda-1)(\lambda-2)(\lambda-3) + \lambda(\lambda-1)(\lambda-2) = \lambda(\lambda-1)(\lambda-2)^2$

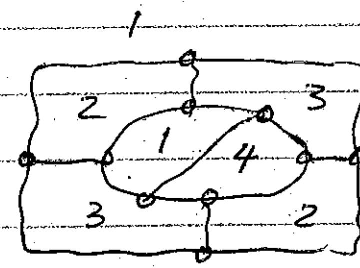
Coloring planar maps.

Def. A *planar map* is any partition of the plane into a finite no. of regions, each of which is bounded by a finite number of curves.

A planar map can be viewed as a planar multi-graph. The edges are the boundary curves and the vertices are the points where 3 or more boundary lines meet. Coloring the regions of a planar map G is equivalent to coloring the vertices of G^*.

The Four Color Problem (Guthrie, 1852)
Can every planar map be legally colored with 4 colors?

Ex.5 Let M = 



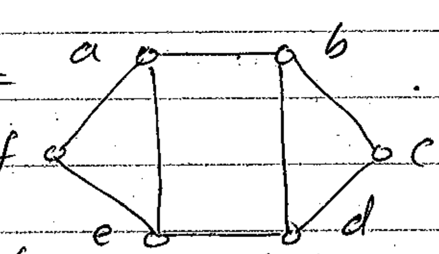
Can M be colored with 4 colors? Yes.

Guthrie was a student of A. De Morgan (well known from DeMorgan's laws). In 1879, Kempe attempted to prove that every planar graph can be legally colored with 4 colors. Unfortunately, there was an error in the proof that could not be fixed. Heawood found the error 1890 & using the same idea proved that 5 colors are sufficient. Finally in 1976, Appel, Haken & Koch proved that the 4 color conjecture is true by reducing it to checking 3 billion situations with the help of computers.

§3. Matchings in graphs

<u>Def.</u> Let G be a graph. A <u>matching</u> of G is any set M of edges in G such that no two edges in M share a common endpoint. A matching M is said to be <u>maximal</u> in G, if there is no match M' of G which properly contains M.

<u>Ex. 1</u> Let $G =$



. Then $M_1 = \{ab, cd\}$, $M_2 = \{ab, de\}$, and $M_3 = \{ab, cd, ef\}$ are matchings of G. M_2 & M_3 are maximal matchings of G but M_1 is not.

Matchings in bipartite graphs are easier to study than matchings in general graphs. If G is bipartite with partite sets X & Y, then by considering X as a set of ladies & Y as a set of gentlemen, a matching of G can be viewed as a set of marriages. (An edge between a vertex in X and a vertex in Y would mean that marriage is acceptable between these two people)

<u>Theorem 7</u> (Hall's Marriage theorem).
Let $G = \langle X \cup Y, E \rangle$ be a bipartite graph with partite sets X & Y and for each $S \subseteq X$, let $N(S) =$ the of all vertices in Y that are adjacent to at least one vertex in S. Then X can be matched with a subset of $Y \iff |N(S)| \geq |S|$ for each subset $S \subseteq X$.

<u>Proof</u>: ($\Rightarrow$) Let $X = \{x_1, \ldots, x_n\}$ and suppose that X can

matched with a subset of Y. Then we can find a a subset, $\{y_1, \dots, y_n\}$ say, of Y such that $M = \{x_1y_1, x_2y_2, \dots, x_ny_n\}$ is a matching of G. Now if $S = \{x_{i_1}, \dots, x_{i_k}\}$ is any subset of X, then $N(S) \supseteq \{y_{i_1}, \dots, y_{i_k}\}$. So $|N(S)| \geq |S|$ for any $S \subseteq X$.

($\Leftarrow$) We will prove this direction of the result by induction on $n = |X|$. If $n=1$ & $|N(S)| \geq |S|$ for each $S \subseteq X$, then the single vertex in X must have an edge to at least one vertex y_1 in Y and this matches X with a subset of Y. Now suppose the result is true for all bipartite graphs with $|X| = n$. Let G be any bipartite graph with $X = \{x_1, \dots, x_{n+1}\}$ & $|N(S)| \geq |S|$ for each $S \subseteq X$. There are two cases

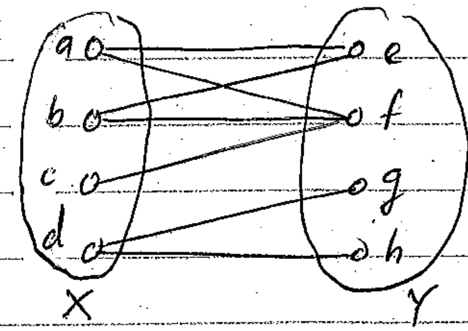
<u>Case (i)</u>: $|N(S)| \geq |S| + 1$ for each proper subset $S \subsetneqq X$. In this case we take any vertex, y_{n+1} say, that is adjacent to x_{n+1} and match it with x_{n+1}. Then for any subset S of $\{x_1, \dots, x_n\}$ we know that $|N(S)| \geq |S| + 1$. So $|N(S) - \{y_{n+1}\}| \geq |S|$ and thus $|N(S) \cap (Y - \{y_{n+1}\})| \geq |S|$ for each $S \subseteq \{x_1, \dots, x_n\}$. By the induction hypothesis, it follows that $\{x_1, \dots, x_n\}$ can be matched with a subset of $Y - \{y_{n+1}\}$. This matching together with the edge $x_{n+1} — y_{n+1}$ will match X with a subset of Y.

<u>Case (ii)</u>: $|N(S_0)| = |S_0|$ for some proper subset $S_0 \subsetneqq X$. In this case, let $X' = X - S_0$ and $Y' = Y - N(S_0)$. Then $|N(S)| \geq |S|$ for each $S \subseteq S_0$ and also

$|N(S')| \geq |S'|$ for each subset $S' \subseteq X'$. So by the Induction hypothesis, we can match S_0 with $N(S_0)$ and X' with a subset of $Y - N(S_0)$. These two matchings will match X with a subset of Y.

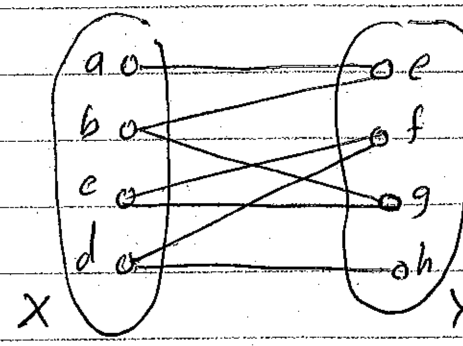
So if the result is true for n, it will be true for $n+1$. Hence by the Principle of mathematical Induction, the result is true for all bipartite graphs.

Ex.1 Let G be the bipartite graph shown on the right. Can X be matched with a subset of Y?



Sol. Since $|N(\{a,b,c\})| = 2 \not\geq 3 = |\{a,b,c\}|$, X cannot be matched with a subset of Y.

Ex.2 Let H be the bipartite graph shown on the right. Can X be matched with a subset of Y?



Sol. If we check all 15 non-empty subsets of X, we will find that $|N(S)| \geq |S|$ for each $S \subseteq X$. So X can be matched with a subset of Y. Actually $M = \{ae, bg, cf, dh\}$ is a matching of all of X with Y.

Stable marriages: Suppose $\{L_1, \ldots, L_n\}$ is a set of n ladies, $\{M_1, \ldots, M_n\}$ is a set of n men, and for each i, $\langle L_i, M_i \rangle$ is a married couple. Now if L_1 prefers M_2 more than her husband M_1 & M_2 prefers L_1 more than his wife L_2, then the situation will be unstable (because L_1 & M_2 will want to get together).

Def. Let M be a matching between $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_n\}$. We say that the matching M is stable if there is no pair $\langle x_i, y_j \rangle$ such that x_i prefers y_j more than her matched partner in M & y_j prefers x_i more than his matched partner in M.

Qu: We are given n ladies and n men. Each lady ranks the men from 1 to n; and each man ranks the ladies from 1 to n. How can we find a stable matching between the n ladies & n men?

Algorithm 2 (Stable Marriage algorithm)

INPUT: The preference tables of n ladies & n men

OUTPUT: A man-optimal set of n stable marriages

1. Each man proposes to his 1st choice at the beginning of the first day.
2. Each lady (who got a proposal) temporarily accepts her most favorable offer at the end of each day; and rejects any other less favorable offers.

3. The men whose proposals were rejected, propose to their next best choice at the beginning of the next day.
4. Steps 2 & 3 are repeated until no proposal is rejected.
5. At this stage the proposing stops and each lady marries her most favorable offer.

Def. A set of stable marriages is said to be ~~man~~ lady-optimal if each ~~man~~ lady gets ~~his~~ her best possible choice among all stable sets of marriages.

Ex. 3 Find (a) the man-optimal & (b) the lady-optimal set of stable marriages for the preference tables below.

MEN PREF	1st	2nd	3rd	4th
A	f	g	e	h
B	f	h	g	e
C	g	h	e	f
D	e	f	h	g

LADIES PREF	1st	2nd	3rd	4th
e	C	A	D	B
f	D	C	B	A
g	D	B	C	A
h	D	A	B	C

(a)

Men Proposing	1st	2nd	3rd	4th	5th day
A	~~f~~	~~g~~	e	e	e
B	f	f	f	~~f~~	h
C	g	g	g	g	g
D	e	e	~~e~~	f	f

man-optimal set
A – e
B – h
C – g
D – f

(b)

Ladies Proposing	1st	2nd day
e	C	C
f	D	D
g	~~D~~	B
h	~~D~~	A

lady-optimal set
e – C
f – D
g – B
h – A