MAD 3305 - GRAPH THEORY

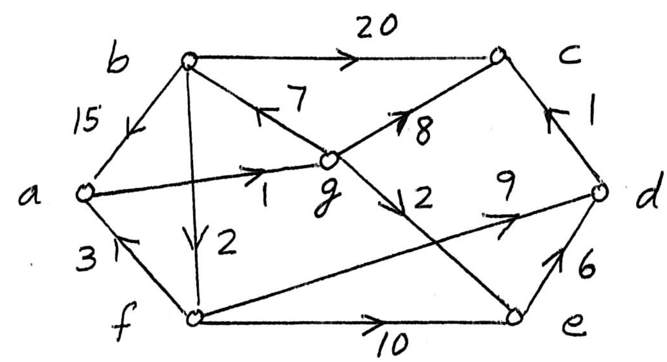
FLORIDA INT'L UNIV.

TEST #1 - FALL 2006

TIME: 75 min.

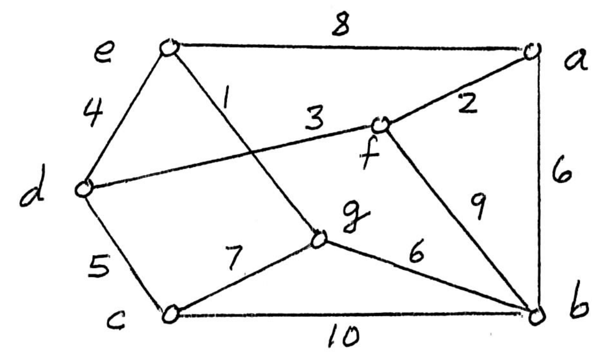
Answer *all* 6 questions. Provide all reasoning and show all working. *An unjustified answer will receive little or no credit.* *BEGIN EACH OF THE 6 QUESTIONS ON A SEPARATE PAGE.*

(15) 1. Find the distances from *b* to each of the other vertices of the graph on the right by using *Dijkstra's Algorithm*.



(20) 2(a) Determine whether or not the sequence 5,4,4,3,3,3 is *graphical*.

(b) Find a minimal spanning tree of the graph on the right by using *Prim's Algorithm* & starting at *b*.



(20) 3. (a) Find the tree corresponding to $\langle 3,1,4,3 \rangle$ via *Prufer's Tree Decoding Algorithm*.

(b) The six characters a,b,c,d,e,f occur with frequencies 25,5,6,13,40,11 respectively. Find an optimal binary coding for these six characters.

(15) 4. (a) Define what is a *connected component* of a disconnected graph G.

(b) Prove that in any tree T=<V,E>, we always have $|E|=|V|-1$.

(15) 5. (a) Define what is the *vertex connectivity* of a graph G.

(b) Let G be a graph with p vertices in which $\delta(G) > (p/2)-1$. Prove that G must be connected.

(15) 6. (a) Define what are the *children* of v in a rooted tree T.

(b) Let T be a ternary tree with p vertices. Prove that
$$h(T) \geq \log_3[(2p+1)/3].$$

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Solutions to Test #1 — Fall 2006

(15) 1.

$L(a)$	$L(b)$	$L(c)$	$L(d)$	$L(e)$	$L(f)$	$L(g)$	T	V_0
∞	0	∞	∞	∞	∞	∞	$\{a,b,c,d,e,f,g\}$	b
15	.	20	∞	∞	$\underline{2}$	∞	$\{a,c,d,e,f,g\}$	f
$\underline{5}$	.	20	11	12	.	∞	$\{a,c,d,e,g\}$	a
.	.	20	11	12	.	$\underline{6}$	$\{c,d,e,g\}$	g
.	.	14	11	$\underline{8}$	.	.	$\{c,d,e\}$	e
.	.	14	$\underline{11}$	.	.	.	$\{c,d\}$	d
.	.	$\underline{12}$	.	.	.	.	$\{c\}$	c
.	.	.	.	.	.	.	$\emptyset$ STOP	

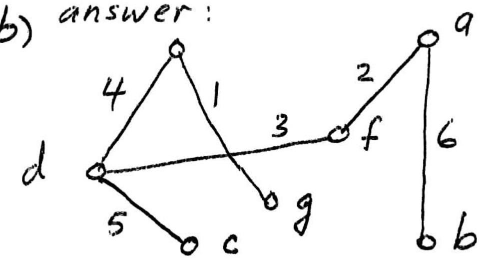
15 0 12 11 8 2 6 ← $d(b,\cdot)$

(6,14) 2(a)

5, 4, 4, 3, 3, 3
3, 3, 2, 2, 2
2, 1, 1, 2
2, 2, 1, 1 → 1, 0, 1
1, 1, 0
0, 0

∴ graphical.

(b) answer:



(b)

$E(T)$	U	a	b	c	d	e	f	g	x_0
		$d(U,\cdot)$							
$\emptyset$	$\{b\}$	$\underline{6}$	.	10	∞	∞	9	6	a
$+\{ab\}$	$\{a,b\}$	.	.	10	∞	8	$\underline{2}$	6	f
$+\{af\}$	$\{a,b,f\}$	.	.	10	$\underline{3}$	8	.	6	d
$+\{fd\}$	$\{a,b,f,d\}$	.	.	5	.	4	.	6	e
$+\{de\}$	$\{a,b,d,e,f\}$	.	.	5	.	.	.	$\underline{1}$	g
$+\{eg\}$	$\{a,b,d,e,f,g\}$	.	.	$\underline{5}$	.	.	.	.	c
$+\{dc\}$	$\{a,b,c,d,e,f,g\}$	STOP							

Note: We could have chosen g at line 1. It would have led to a different spanning tree with the same total weight of 21.

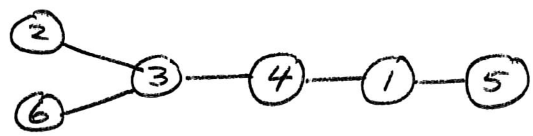
(10,10) 3(a) $\underset{\sim}{s} = \langle 3,1,4,3 \rangle$, so $p = |\underset{\sim}{s}| + 2 = 6$

X	Edges x needs when i=1	i=2	i=3	i=4	
1	2	2 →	1	0	0
2	1 →	0	0	0	0
3	3 →	2	2	2 →	1
4	2	2	2 →	1 →	0
5	1	1 →	0	0	0
6	1	1	1	1	1

smallest vertex which needs only 1 more edge

i	j(i)	s(i)
1	2 —	3
2	5 —	1
3	1 —	4
4	4 —	3
plus	3 —	6

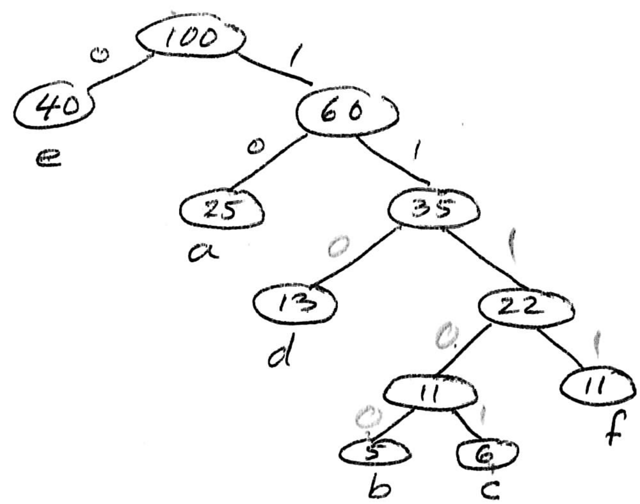
Ans: T =



(b) Ans:

Character	a	b	c	d	e	f
Frequency	25	5	6	13	40	11
Code	10	11100	11101	110	0	1111

5, 6, 11, 13, 25, 40
11, 11, 13, 25, 40
13, 22, 25, 40
25, 35, 40
40, 60
100.



4(a) A connected component of G is any connected subgraph H of G which is maximal (in the sense that there are no paths from a vertex of H to a vertex outside of H). You could also say: Let $u \in V(G)$ & $G(u)$ = the subgraph of G containing all the edges & vertices that are accessible from u. Then $G(u)$ will be a connected component of G.

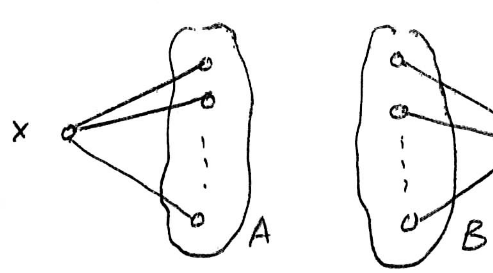
4.(b) We will prove the result by induction on $|V|$. For $|V|=1$, we have $T = K_1$, so $|E| = 0 = |V|-1$. So the result is true for $|V|=1$. Now suppose the result is true for all trees with $|V| \le k$. Let T be any tree with $k+1$ vertices. and e be any edge in T. Then $T-\{e\} = T_1 \cup T_2$ where T_1 & T_2 are two disjoint trees. So

$$|E(T)| = |E(T_1)| + |E(T_2)| + 1$$
$$= (|V(T_1)|-1) + (|V(T_2)|-1) + 1 = |V(T)| - 1$$

So if the result is true for all trees with $|V| \le k$, it will be true for all trees with $|V| = k+1$. Hence by the 2nd Principle of Math Ind., the result is true for all trees.

5 (a) The vertex connectivity of G is the smallest no. of vertices whose removal will disconnect G or reduce it to K_1.

(b) Suppose G is not connected. Then we can find two vertices x and y such that there is no path from x to y in G. Let A = set of all vertices in G that are adj. to x and B = set of all vertices in G that are adj. to y. Then $A \cap B = \emptyset$ because there is no path from x to y and $A \cup B \subseteq V(G) - \{x, y\}$. So $|A \cup B| \le p-2$.



Since $A \cap B = \emptyset$, it follows that $|A| + |B| \le p-2$. Hence $|A| \le \frac{p-2}{2}$ or $|B| \le \frac{p-2}{2}$. So $\deg(x) = |A| \le \frac{p}{2} - 1$ or $\deg(y) = |B| \le \frac{p}{2} - 1$. But this contradicts the fact that $\delta(G) > \frac{p}{2} - 1$. Hence G must be connected.

6 (a) The children of the vertex v in the rooted tree T are all the vertices at level $k+1$ which are adjacent to v. Here we assume that v is at level k in T.

(b) Since T is a ternary tree there will be at most

1 vertex at level 0

3 vertices at level 1

3^2 " " " 2

...

and 3^i vertices at level i. So if $k = h(T)$

$$p \leq 1 + 3 + 3^2 + \cdots + 3^k = \frac{3^{k+1} - 1}{3 - 1}$$

$\therefore \quad 2p \leq 3^{k+1} - 1. \quad \therefore \quad 2p+1 \leq 3 . 3^k$

Thus $\quad 3^k \geq (2p+1)/3.$

$\therefore \quad k = h(T) \geq \log_3 \left(\frac{2p+1}{3}\right).$