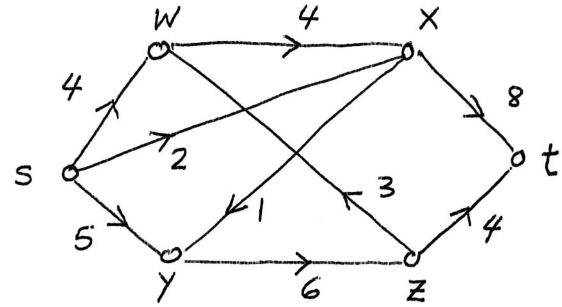
MAD 3305 - GRAPH THEORY — FLORIDA INT'L UNIV.

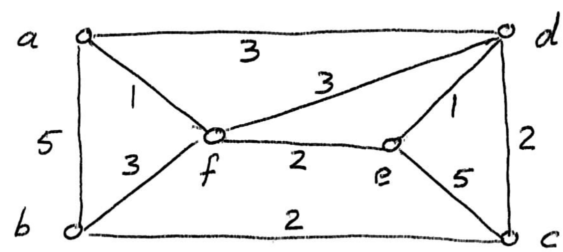
TEST #2 - FALL 2006 — TIME: 75 min.

Answer all 6 questions. Provide all reasoning and show all working. An unjustified answer or failure to follow instructions will result in little credit. BEGIN EACH QUESTION ON A SEPARATE PAGE.

(15) 1. Find a maximal flow f^* in the network on the right by using the Ford-Fulkerson algorithm. Also find the source-separating set of vertices S^* corresp. to f^*.

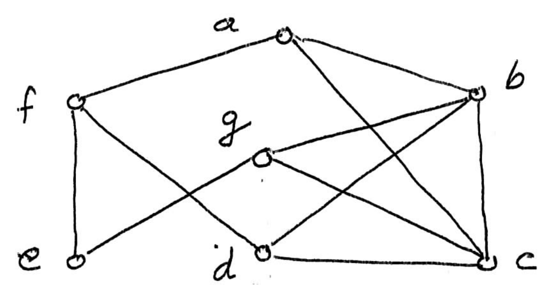


(15) 2. (a) Find a minimum postman walk of the graph on the right by using the Postman algorithm.
(b) What is the total length of your walk?

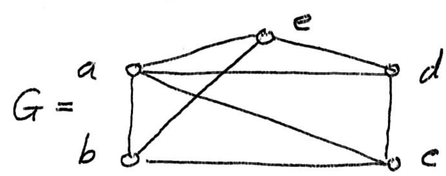


(15) 3. Using the DMP planarity algorithm, determine whether or not the graph on the right is planar.

(Show the embeddings for each step of the algorithm)



(25) 4. (a) Find the Chromatic Polynomial of the graph G on the right.



(b) Prove that every region in a maximal planar graph with $p>2$ is bounded by exactly 3 edges.

(15) 5. (a) Define what is an *Ore-type graph*.
(b) Suppose $v_1, v_2, \ldots, v_n$ is a maximal path in an Ore-type graph G and v_1&v_n are non-adjacent. Prove that $v_1, v_2, \ldots, v_n, v_1$ can always be rearranged to form a cycle in G.

(15) 6. (a) Define what is a *regular polyhedron*.
(b) Let H be a polyhedron which has no triangular face. If H has p vertices and q edges, prove that $q \leq 2(p-2)$.
[You may use any theorem proved in class for Qu.#6]

MAD 3305 - Graph Theory

Solutions to Test #2

Florida Int'l Univ.

Fall 2006

1\. 1st Aug. semi-path:

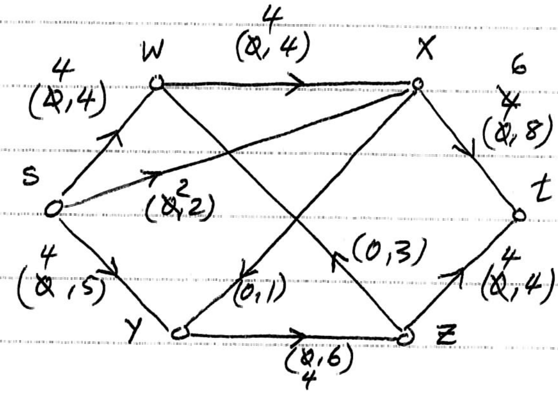
$s \xrightarrow{(0,4)} w \xrightarrow{(0,4)} x \xrightarrow{(0,8)} t$

slacks 4 4 8 $\lambda = 4$

2nd Aug. semi-path:

$s \xrightarrow{(0,2)} x \xrightarrow{(4,8)} t$

slacks 2 4 $\lambda = 2$



3rd. Aug. semi-path: $s \xrightarrow{(0,5)} y \xrightarrow{(0,6)} z \xrightarrow{(0,4)} t$

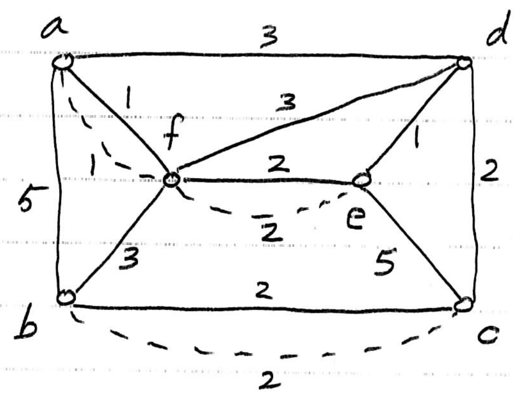
slacks 5 6 4 $\lambda = 4$

$S^* = \{s, y, z, w\}$. $c(S^*) = 4+2+4 = 10$

$F(f^*) = (4+2+4) - 0 = 10$ ✓

2\. Odd vertices: $\{a, b, c, e\}$

$d(\cdot,\cdot)$	a	b	c	e
a	·	4	5	3
b		·	2	5
c			·	3
e				·



$\{a,b\} + \{c,e\}$ $\{a,c\} + \{b,e\}$ $\{a,e\} + \{b,c\}$

$4 + 3 = 7$ $5 + 5 = 10$ $3 + 2 = 5$ ✓ pick this partition

Minimum postman walk is

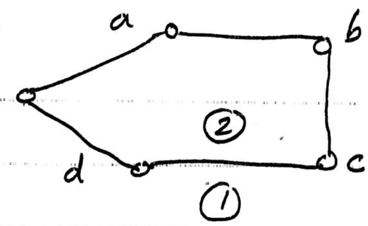
$a \xrightarrow{3} d \xrightarrow{2} c \xrightarrow{2} b \xrightarrow{2} c \xrightarrow{5} e \xrightarrow{1} d \xrightarrow{3} f \xrightarrow{2} e \xrightarrow{2} f \xrightarrow{3} b \xrightarrow{5} a \xrightarrow{1} f \xrightarrow{1} a$

Total length of the min. postman walk

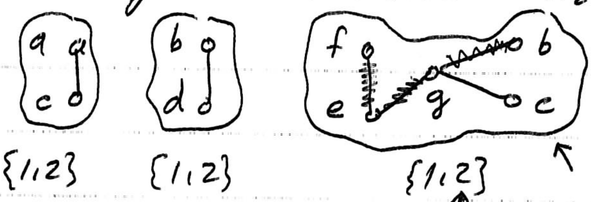
$= 3+2+2+2+5+1+3+2+2+3+5+1+1 = 32$

Check: $w(G) + 5 = 27 + 5 = 32$ ✓

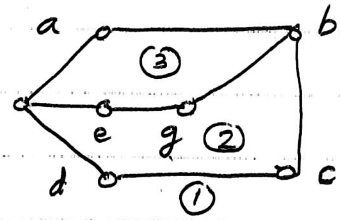
3. $H_1 =$

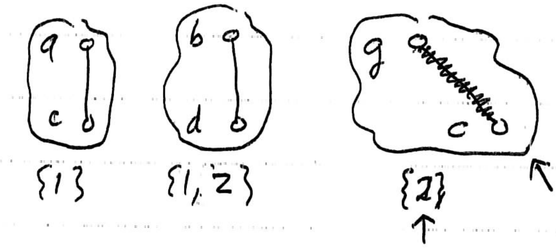


Segments relative to H_i

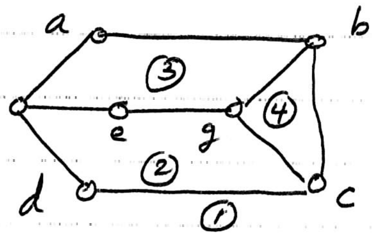


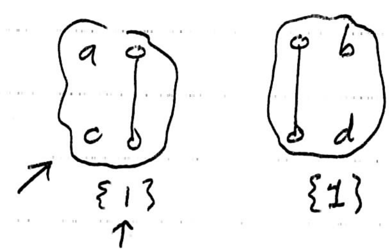
$H_2 =$



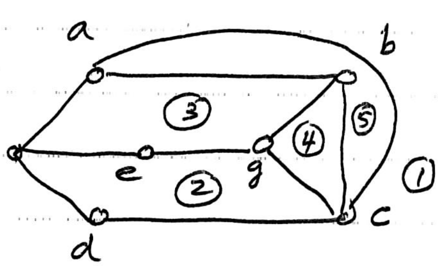


$H_3 =$



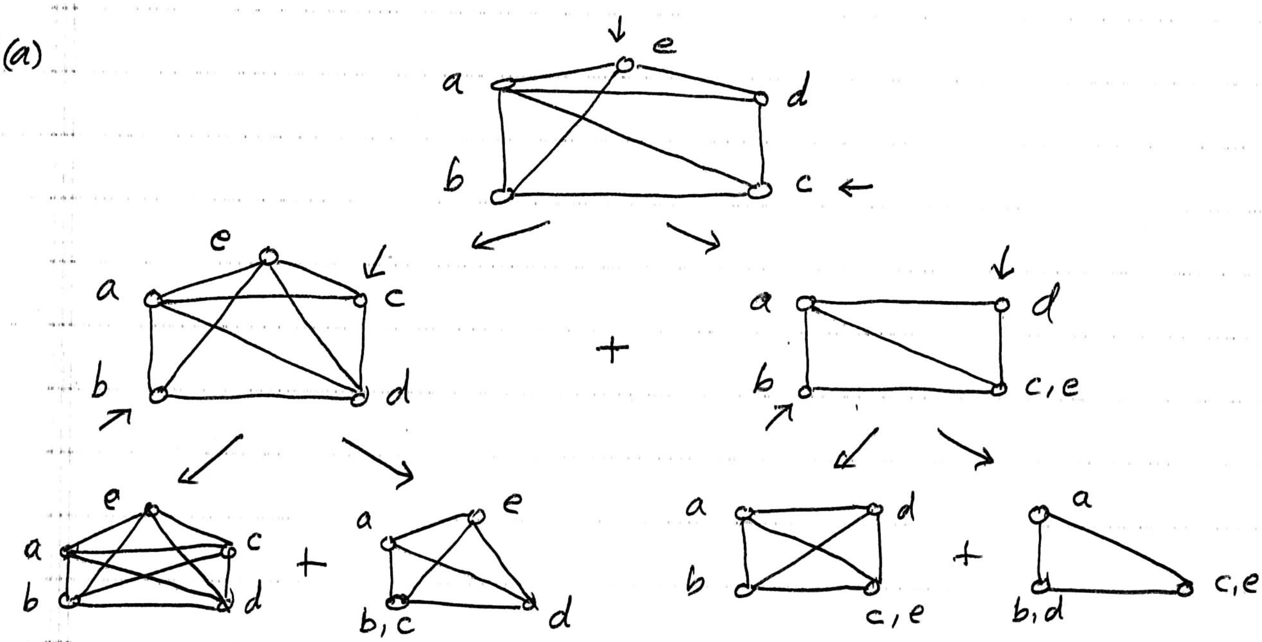


$H_4 =$



$\emptyset$ ← NONPLANAR

4(a)



$$P_G(\lambda) = P_{K_5}(\lambda) + 2P_{K_4}(\lambda) + P_{K_3}(\lambda) = \lambda^{(5)} + 2\lambda^{(4)} + \lambda^{(3)}$$

$$= \lambda(\lambda-1)(\lambda-2)\left[(\lambda-3)(\lambda-4) + 2(\lambda-3) + 1\right]$$

$$= \lambda(\lambda-1)(\lambda-2)(\lambda^2-5\lambda+7).$$

4(b) Suppose G is a maximal planar graph and one of the regions R in a planar embedding of G is bounded by 4 or more edges. Then the boundary of this region will be a cycle $v_1, v_2, v_3, \ldots, v_k, v_1$ where $k \geq 4$. There are two cases

Case (i) : $v_1 v_3 \notin E(G)$. In this case we can add a new edge $v_1 v_3$ to G inside the region R without intersecting any other edge. But this contradicts the fact that G was a maximal planar graph.

Case (ii) : $v_1 v_3 \in E(G)$. In this case the edge $v_2 v_k$ cannot be in G otherwise it would intersect $v_1 v_3$ in the embedding. So we can add a new edge $v_2 v_k$ inside the region R without intersecting any other edge and again this contradicts the fact that G was maximal planar.

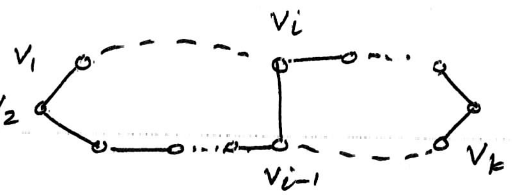
So in both cases we got a contradiction. Hence every region of G will be bounded by at most 3 edges. Since G is a graph, there are no regions with 2 or 1 boundaries. Hence every region is bounded by exactly 3 edges.

5(a) An Ore-type graph is any graph with p vertices such that $\deg(x) + \deg(y) \geq p$ for any two non-adj. vertices x & y.

(b) Suppose $v_1, v_2, \ldots, v_n$ is a maximal path in an Ore-type graph G and v_1 & v_n are non-adjacent. Then we can find an i such that $v_1 v_i$ & $v_{i-1} v_k$ are edges in G. Indeed suppose there is no such i. Then every time v_1 is adjacent to a vertex, v_k will be non-adjacent to the preceding vertex. (Remember v_1 & v_k can only be adjacent to vertices in $\{v_2, \ldots, v_{k-1}\}$ because the path $v_1, \ldots, v_n$ was a maximal path.) So $\deg(v_k) \leq (p-1) - \deg(v_1)$

$\deg(v_1) + \deg(v_k) \leq \deg(v_1) + (p-1) - \deg(v_1) \leq p-1$ contradicting the fact that G was an Ore-type graph.

5 (b)



Now if we look at the seq $v_1, v_2, \ldots, v_{i-1}, v_k, v_{k-1}, \ldots v_{i+1}, v_i, v_1$ we see that it will be a cycle in G.

6 (a) A regular polyhedron is a simple polyhedron in which all the faces are the same fixed regular polygon and in which each vertex subtend the same solid angle.

(b) Suppose H has no triangular face. Then each of the faces $A_1, \ldots, A_r$ of H will be bounded by at least 4 edges. Now since each edge is in two faces,

$$2q = \text{number of edges counted using the faces}$$
$$= e(A_1) + \cdots + e(A_r) \geq \underbrace{4 + \cdots + 4}_{r \text{ times}} = 4r$$

$\therefore \quad 2q \geq 4r.$

So $\quad q \geq 2r.$ But $r = q + 2 - p$. So

$q \geq 2(q+2-p)$. Hence $2(p-2) \geq q$

$\therefore \quad q \leq 2 \cdot (p-2).$