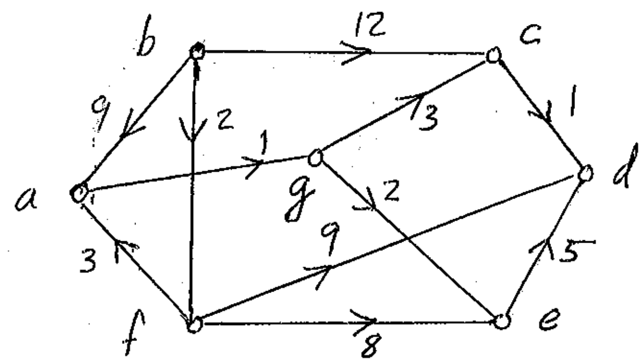
MAD 3305 - GRAPH THEORY FLORIDA INT'L UNIV.
TEST #1 - FALL 2014 TIME: 75 min.

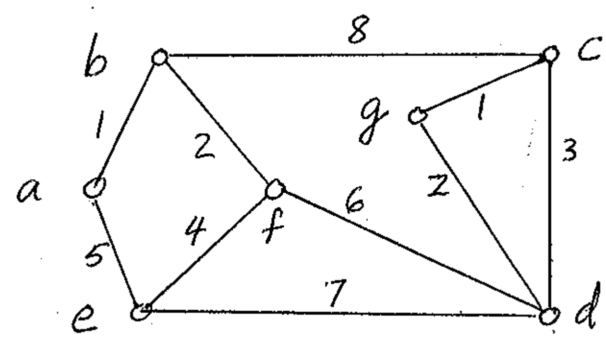
Answer *all 6* questions. ***No Calculators or Cellphones are allowed.*** *An unjustified answer will receive little or no credit. BEGIN EACH OF THE 6 QUESTIONS ON A SEPARATE PAGE.*

(15) 1. Find the ***distances*** from ***b*** to each of the other vertices of the graph on the right by using *Dijkstra's Algorithm.*



(20) 2(a) Find a ***graph*** with degree sequence 4,3,3,2,2 by using the *Graphical Sequence Algorithm.*

(b) For the graph on the right, find a ***minimal spanning tree*** by using *Kruskal's Algorithm* (show the partitions as you go along).



(20) 3(a) Find the ***tree*** corresponding to the sequence ⟨3,2,3,5⟩ via *Prufer's Tree Decoding Algorithm.*

(b) The five characters a, b, c, d, e occur with frequencies 1, 2, 3, 5, 7 respectively. Find an ***optimal binary coding*** for these five characters and the ***weighted-path length*** of your coding by using *Huffman's algorithm.*

(15) 4(a) Define what is the ***edge-connectivity*** $k_E(G)$ of a graph G.

(b) Let *G* be a digraph on the vertices $V = \{1,2,3, \ldots, p\}$. Prove that the ***number of directed walks of length n*** from i to j is given by $(A^n)[i, j]$, where A = the adjacency matrix of *G*.

(15) 5(a) Define what is the ***distance from u to v*** in a weighted digraph *G*.

(b) Prove that if G is a graph with $\delta(G) \geq (p-1)/2$, then G is ***connected.***

(15) 6(a) Define what is a ***legal flow*** in a network $N = \langle G, s, t, c \rangle$.

(b) Let T be a 4-ary tree with 3,333 vertices. Prove that $h(T) \geq \log_4(2{,}500)$.

MAD 3305 - Graph Theory — Florida Int'l Univ.

Solutions to Test #1 — Fall 2014

1

$L(a)$	$L(b)$	$L(c)$	$L(d)$	$L(e)$	$L(f)$	$L(g)$	T	i	x_0
∞	0	∞	∞	∞	∞	∞	$\{a,b,c,d,e,f,g\}$	0	b
9	,	12	∞	∞	2	∞	$\{a,c,d,e,f,g\}$	1	f
5	,	12	11	10	,	∞	$\{a,c,d,e,g\}$	2	a
,	,	12	11	10	,	6	$\{c,d,e,g\}$	3	g
,	,	9	11	8	,	,	$\{c,d,e\}$	4	e
,	,	9	11	,	,	,	$\{c,d\}$	5	c
,	,	,	10	,	,	,	$\{d\}$	6	d
,	,	,	,	,	,	,	$\emptyset$	STOP	

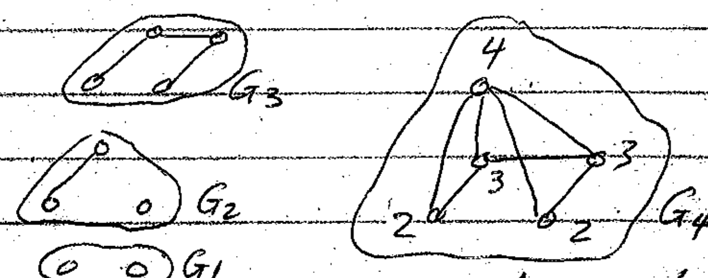
$d(b,\cdot) = 5 \quad 0 \quad 9 \quad 10 \quad 8 \quad 2 \quad 6$

2(a) 4, 3, 3, 2, 2

2, 2, 1, 1

1, 0, 1 → 1, 1, 0

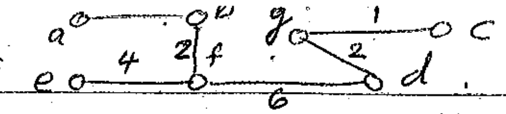
0, 0



$e_9 = de$, $e_{10} = bc$

2(b) $e_1 = ab$, $e_2 = cg$, $e_3 = bf$, $e_4 = dg$, $e_5 = cd$, $e_6 = ef$, $e_7 = ae$, $e_8 = df$,

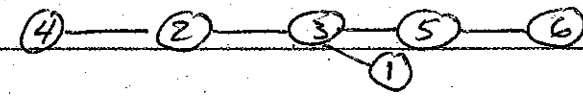
$E(T)$	Parts of the Partition	i	endpoints(e_i)
$\emptyset$	$\{a\}\{b\}\{c\}\{d\}\{e\}\{f\}\{g\}$	1	$\{a,b\}$
$\{ab\}$	$\{a,b\}\{c\}\{d\}\{e\}\{f\}\{g\}$	2	$\{c,g\}$
$\{ab,cg\}$	$\{a,b\}\{c,g\}\{d\}\{e\}\{f\}$	3	$\{b,f\}$
$\{ab,cg,bf\}$	$\{a,b,f\}\{c,g\}\{d\}\{e\}$	4	$\{d,g\}$
$\{ab,cg,bf,dg\}$	$\{a,b,f\}\{c,d,g\}\{e\}$	5	$\{c,d\}$
don't add cd	—	6	$\{e,f\}$
$\{ab,cg,bf,dg,ef\}$	$\{a,b,e,f\}\{c,d,g\}$	7	$\{a,e\}$
don't add ae	—	8	$\{d,f\}$
$\{ab,cg,bf,dg,ef,df\}$	$\{a,b,c,d,e,f,g\}$	STOP	

2(b) Answer = 

The tree will be on $\{1,2,\dots,6\}$ bec. Length of $\underset{\sim}{s} = 4 \Rightarrow p=6$

3(a)

$d_i(1)$	$d_i(2)$	$d_i(3)$	$d_i(4)$	$d_i(5)$	$d_i(6)$	i	$l(i)$	$s(i)$
1	2	3	1	2	1	1	1	3
0	2	2	1	2	1	2	4	2
0	1	2	0	2	1	3	2	3
0	0	1	0	2	1	4	3	5
0	0	0	0	1	1	5	add 5–6	

Ans: = 

(b)

Characters	a	b	c	d	e
Frequencies	1	2	3	5	7
Coding	1110	1111	110	10	0
Length of code	4	4	3	2	1

WPL = 4+8+9+10+7
= 38.

$\underbrace{1, 2}, 3, 5, 7$

$\underbrace{3, 3}, 5, 7$

$\underbrace{5, 6}, 7$

$\underbrace{7, 11}$

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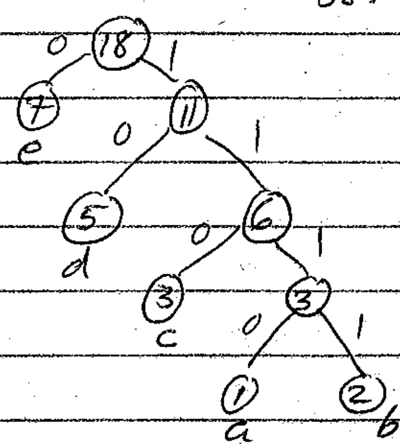

4(a) The edge connectivity of G is defined by $k_E(G)$ = min. no. of edges whose removal are needed to disconnect G or reduce it to K_1.

(b) Let G be a directed graph on the vertices $V=\{1,\dots,p\}$.
We will prove the result by induction on p. For $n=1$, we have
no. of directed walks from i to j = no. of directed edges from i to j = $A^1[i,j]$
So the result is true for $n=1$. Suppose that the result is true for all i & j for walks of length n. Then

$$\begin{pmatrix}\text{No. of directed}\\ \text{walks of length}\\ n+1 \text{ from } i \text{ to } j\end{pmatrix} = \sum_{k=1}^{n} \begin{pmatrix}\text{no. of directed}\\ \text{walks of length}\\ n \text{ from } i \text{ to } k\end{pmatrix} \cdot \begin{pmatrix}\text{no. of directed}\\ \text{walks of length}\\ 1 \text{ from } k \text{ to } j\end{pmatrix}$$

$$= \sum_{k=1}^{p} A^n[i,k] \cdot A^1[k,j] = (A^{n+1})[i,j].$$

4(b) So if the result is true for n, it will be true for $n+1$.
By the PMI, it follows that the result is true for all $n, i \,\&\, j$.

5(a) $d(u,v) = \begin{cases} \text{length of the shortest directed path from } u \text{ to } v \text{ in } G \\ \infty, \text{ if there is no directed path from } u \text{ to } v \text{ in } G. \end{cases}$

(b) It will suffice to prove the contrapositive of the given statement, namely, "If G is a disconnected graph with p vertices, then $\delta(G) < \frac{(p-1)}{2}$."
Suppose G is a disconnected graph with p vertices. Then we can find two vertices x & y such that there is no path from x to y in G. Let A = set of vertices adjacent to x and B = set of vertices adjacent to y. Then $A \cap B = \emptyset$ and $|A \cup B| \leq p-2$ (because $x \notin A \cup B$ & $y \notin A \cup B$) So $|A| + |B| \leq p-2$.
$\therefore$ $|A| \leq (p-2)/2$ or $|B| \leq (p-2)/2$. But $|A| = \deg(x)$ and $|B| = \deg(y)$. So $\deg(x) \leq (p-2)/2 < (p-1)/2$ or $\deg(y) \leq (p-2)/2 < (p-1)/2$. $\therefore$ $\delta(G) < (p-1)/2$ & we are done.

6(a) A legal flow in $N = \langle G, s, t, c \rangle$ is any function $f : E(G) \to \mathbb{R}^{\#}$ such that $f(e) \leq c(e)$ for each $e \in E(G)$ and
$\sum_{e \in In(v)} f(e) = \sum_{e \in Out(v)} f(e)$ for each $v \in V(G) - \{s, t\}$.

(b) Let $k = h(T)$. Since T is a 4-ary tree, there can be at most 4^0 vertices at level 0, 4^1 vertices at level 1, 4^2 vertices at level 2, ..., & 4^k vertices at level k.

So $3{,}333 \leq 1 + 4^1 + 4^2 + \cdots + 4^k = \dfrac{4^{k+1} - 1}{4-1}$

$\therefore$ $3(3{,}333) \leq 4^{k+1} - 1$

$\therefore$ $9{,}999 + 1 \leq 4^{k+1} \Rightarrow (10{,}000)/4 \leq 4^k$.

So $2{,}500 \leq 4^k$ & hence $\log_4(2{,}500) \leq k$

Thus $h(T) = k \geq \log_4(2{,}500)$.