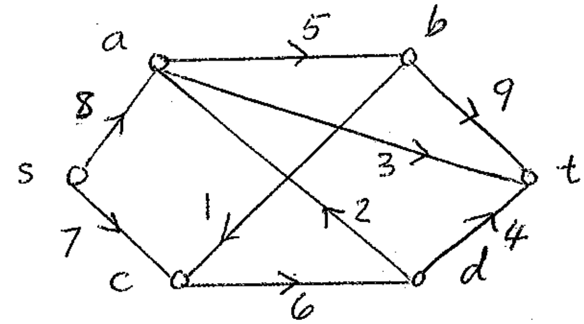
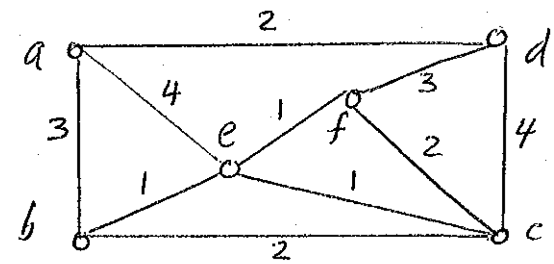
MAD 3305 - GRAPH THEORY | FLORIDA INT'L UNIV.
TEST #2 - FALL 2014 | TIME: 75 min.

Answer *all 6* questions. ***No Calculators or Cellphones are allowed.*** *An unjustified answer will receive little or no credit. BEGIN EACH OF THE 6 QUESTIONS ON A SEPARATE PAGE.*

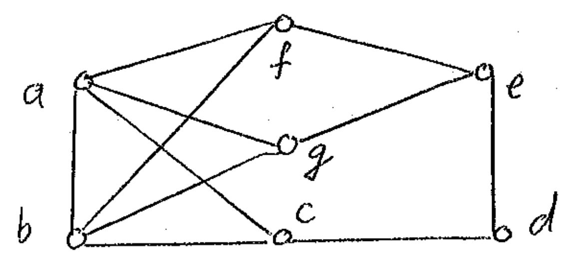
(15) 1. Find a maximal flow f^* in the network on the right by using the *Ford-Fulkerson Algorithm*. Also find the source-separating set of vertices S^* corresponding to f^*.



(15) 2. Find a *minimum postman walk* of the graph on the right by using the *Postman Algorithm*; and find *the total length* of your minimum postman walk?

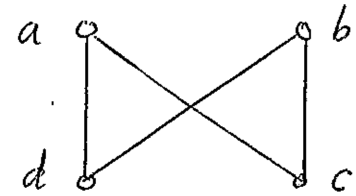


(20) 3. Determine whether or not the graph on the right is planar by using the *DMP Planarity Algorithm*. [*Show the embeddings for each step of the algorithm.*]



(17) 4(a) Find $P_G(\lambda)$ for the graph G on the right by using the Chromatic Polynomial Algorithm.

(b) Prove $P_G(\lambda) = P_{G\cup\{ab\}}(\lambda) + P_{G\circ\{ab\}}(\lambda)$, if a and b are non-adjacent in G.



(15) 5(a) Define what is a *minimum postman walk* of a graph *G*.

(b) Use the *Euler Circuit Theorem* to prove that a connected graph has an open Euler trail if and only if it has exactly two vertices of odd degree.

(18) 6(a) Define what is the *dual* of a planar graph G with respect to the planar embedding E.

(b) Let G be a *planar graph* with p vertices and q edges. If $p \geq 4$ and G has no cycles of length < 6, prove that $2q \leq 3p - 6$.

[*You may use any theorem proved in class for Qu.#6*]

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Solutions to Test #2 — Fall 2014

1(a) 1st Aug. semi-path:

$s \xrightarrow{0,8} a \xrightarrow{0,5} b \xrightarrow{0,9} t$

slacks: 8 5 9, $\mu_1 = 5$

2nd Aug. semi-path:

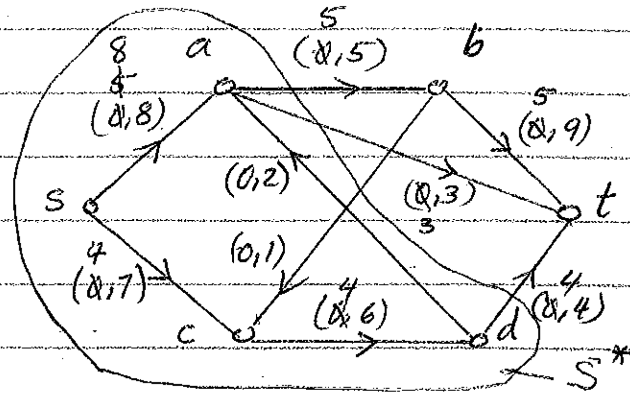
$s \xrightarrow{(5,8)} a \overset{(0,3)}{\text{———}} t$

slacks: 3 3, $\mu_2 = 3$

3rd Aug. semi-path:

$s \xrightarrow{(0,7)} c \xrightarrow{(0,6)} d \xrightarrow{(0,4)} t$

slacks: 7 6 4, $\mu_3 = 4$



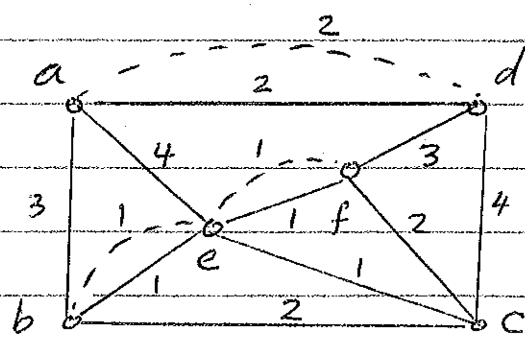
$F(f^*)$ = net flow out of s $= 8+4 = 12$.

(b) $S^* = \{u \in V(G) :$ there is an aug. semi-path from s to $u\} = \{s, c, d, a\}$.

$c(S^*)$ = sum of outward capacities of S^* $= 5+3+4 = 12$ ✓.

2(a) Odd vertices are: a, b, d, f.

dist.	a	b	d	f
a	.	3	2	5
b	.	.	5	2
d	.	.	.	3

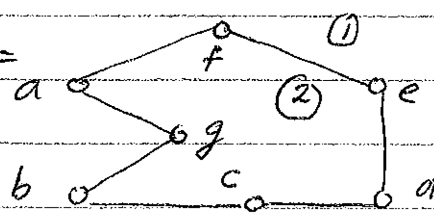


∴ $\{a,b\}+\{d,f\}$: $3+3=6$ $\{a,d\}+\{b,f\}$: $2+2=4$ $\{a,f\}+\{b,d\}$: $5+5=10$

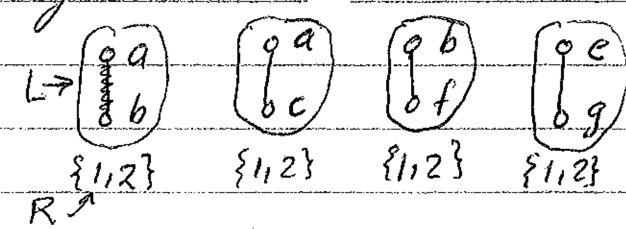
(b) Min. Postman walk is: $a \overset{2}{—} d \overset{2}{—} a \overset{3}{—} b \overset{1}{—} e \overset{1}{—} f \overset{1}{—} e \overset{1}{—} b \overset{2}{—} c \overset{4}{—} d \overset{3}{—} f \overset{2}{—} c \overset{1}{—} e \overset{4}{—} a$.

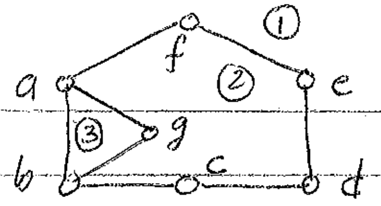
Total length $= 2+2+3+1+1+1+1+2+4+3+2+1+4 = 27$

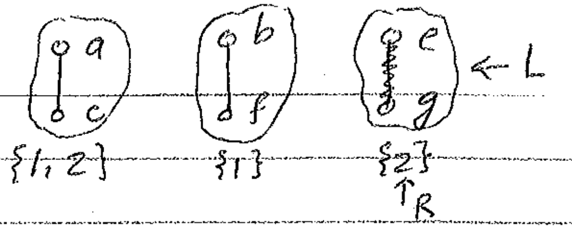
3. H_1 =

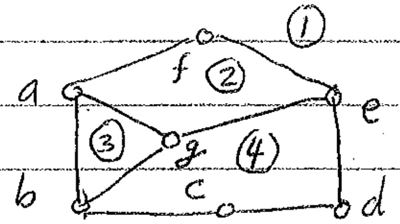


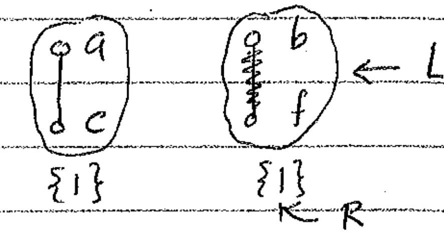
Segments of G relative to H_i:

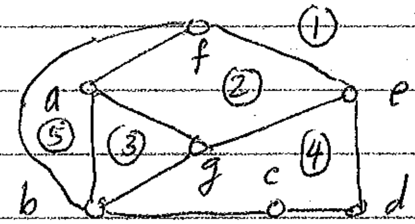


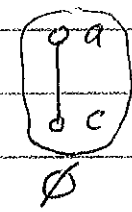
3 $H_2 =$ 



$H_3 =$ 

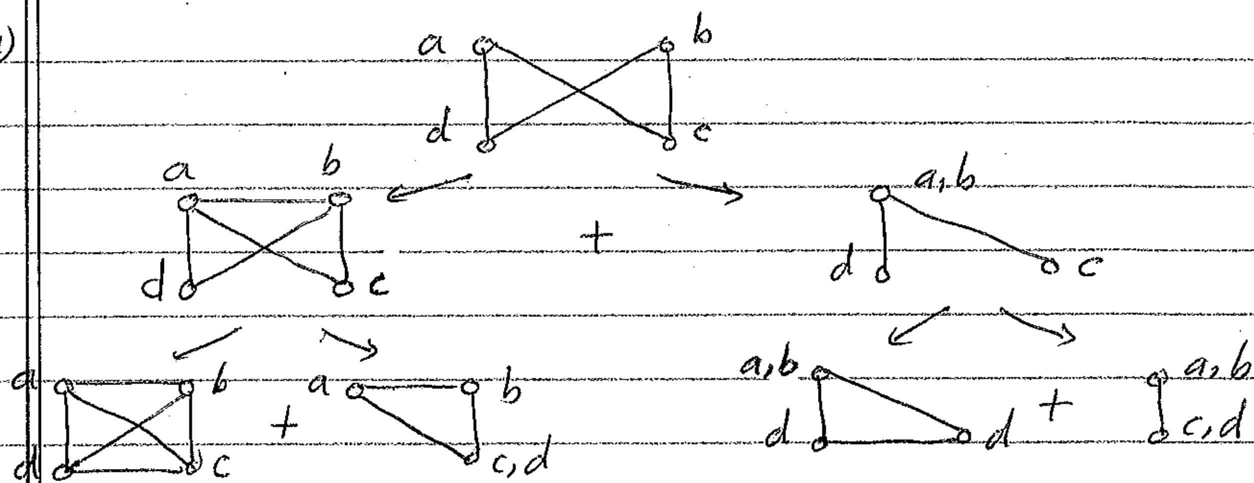


$H_4 =$ 



$\therefore$ Graph is NON-PLANAR.

4 (a)



$$P_G(\lambda) = P_{K_4}(\lambda) + 2P_{K_3}(\lambda) + P_{K_2}(\lambda) = \lambda(\lambda-1)(\lambda-2)(\lambda-3) + 2\lambda(\lambda-1)(\lambda-2) + \lambda(\lambda-1) = \lambda(\lambda-1)\left[(\lambda-2)(\lambda-3)+2(\lambda-2)+1\right]$$
$$= \lambda(\lambda-1)(\lambda^2-3\lambda+3).$$

(b) $P_G(\lambda)$ = No. of ways of legally coloring G with λ colors available at our disposal

= No. of ways of coloring G with a & b colored differently + No. of ways of coloring G with a & colored the same

$= P_{G\cup\{a,b\}}(\lambda) + P_{G\circ\{a,b\}}$ because a & b are non-adjacent vertices in G.

5(a) A minimum Postman walk is a closed walk which includes each edge of G and is of minimum total length.

(b) ($\Rightarrow$) Suppose G has an open Euler trail, say $v_0, e_1, v_1, e_2, v_2, \ldots, v_{q-1}, e_q, v_q$. If we add a new edge e' from v_q to v_0 to G we will get a connected multi-graph G' with an Euler circuit $v_0, e_1, v_1, \ldots, e_q, v_q, e', v_0$. By Euler's Circuit Theorem, it follows that each vertex of G is of even degree. So in $G = G' - \{e'\}$, v_0 & v_q must be of odd degrees.

($\Leftarrow$) Suppose G has exactly two vertices, v_i & v_j say, of odd degrees. Let e' be a new edge from v_j to v_i. Then $G' = G \cup \{e'\}$ will be a connected multi-graph with all vertices of even degree. So by Euler Circuit Theorem G' will have an Euler circuit Q. If we remove e' from Q, we will get an open Euler trail of G.

6(a) $V(G^*_{\mathcal{E}})$ = set of all the regions into which $\mathcal{E}$ partitions $\mathbb{R}^2$. Also if e is an edge of $\mathcal{E}$ between the regions R_1 & R_2, then we get an edge of $E(G^*_{\mathcal{E}})$ between R_1 & R_2.

(b). Suppose $p \geq 4$ & G has no cycles of length < 6. Let $\mathcal{E}$ be a planar embedding of G. Now if $\mathcal{E}$ has no cycles, then G must be a forest and so $q \leq p-1$. Thus $2q \leq 2p-2 = (3p-p)-2 = 3p-(p+2) \leq 3p-6$ because $p \geq 4$.
And if $\mathcal{E}$ has at least one cycle, let $A_1, \ldots, A_r$ be the regions into which $\mathcal{E}$ partitions $\mathbb{R}^2$, where $r = r(G)$. Since each region is bounded by a cycle (which must have at least 6 edges) and each edge is counted twice

$$2q = \text{no. of edges counted by summing over the regions}$$
$$= e(A_1) + e(A_2) + \cdots + e(A_r) \geq \underbrace{(6+6+\cdots+6)}_{r\text{ times}} = 6 \cdot r.$$

$\therefore$ $2q \geq 6r$. So $q \geq 3r$. Since $r \geq q+2-p$, we get $q \geq 3(q+2-p)$ $\therefore$ $3p-6 \geq 2q$. So $2q \leq 3p-6$ in both cases.