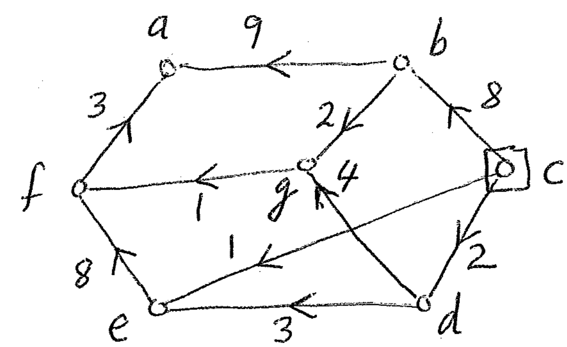
MAD 3305 - GRAPH THEORY
TEST #1 - FALL 2015

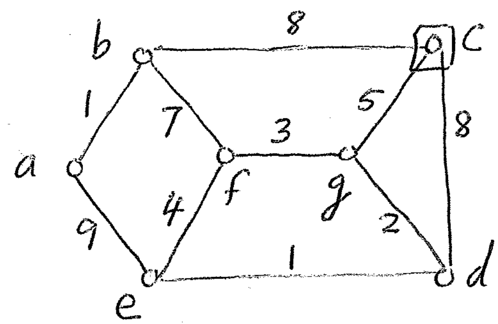
FLORIDA INT'L UNIV.
TIME: 75 min.

Answer *all 6* questions. **No Calculators or Cellphones are allowed.** *An unjustified answer will receive little or no credit. BEGIN EACH OF THE 6 QUESTIONS ON A SEPARATE PAGE.*

(15) 1. Find the *distances* from c to each of the other vertices of the graph on the right by using *Dijkstra's Algorithm.*



(20) 2(a) Find a *graph* with degree sequence $\langle 4, 2, 2, 2, 2\rangle$ by using the *Graphical Sequence Algorithm.*
(b) For the graph on the right, find a *minimal spanning tree* by using *Prim's Algorithm* and starting at c.



(20) 3(a) Find the *tree* corresponding to the sequence $\langle 6, 2, 4, 6\rangle$ via *Prufer's Tree Decoding Algorithm.*
(b) The five characters a, b, c, d, e occur with frequencies *2, 2, 3, 6, 7* respectively. Find an *optimal binary coding* for these five characters and the *weighted-path length* of your coding by using *Huffman's algorithm.*

(15) 4(a) Define what is the *adjacency matrix* A_G of a graph with $V(G)=\{1,2,3, \ldots , p\}$.
(b) Let G be a graph with $V(G) = \{1,2,3, \ldots , p\}$. Prove that the *number of walks of length n* from i to j is given by $(A_G)^n\,[i, j]$.

(15) 5(a) Define what is the *distance from u to v* in a weighted *digraph G.*
(b) Prove that if G is a disconnected graph, then G^C will always be *connected.*

(15) 6(a) Define what is a *legal flow* in a network $N = \langle G, s, t, c\rangle$.
(b) A certain tree T has 50 vertices. Five are of degree 4, ten are of degree 3, and the rest are of degree 1 or 2. How any vertices of degree 2 does T have?
[*You may use any theorem that was proved in class to answer Question #6.*]

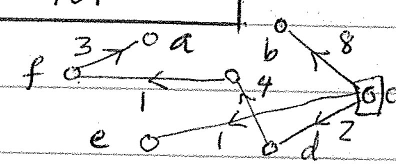
MAD 3305 - Graph Theory
Solutions to Test #1

Florida International Univ.
Fall 2015

1.

L(a)	L(b)	L(c)	L(d)	L(e)	L(f)	L(g)	T	i	v_0
∞	∞	0	∞	∞	∞	∞	{a,b,c,d,e,f,g}	0	c
∞	8	.	2	1	∞	∞	{a,b,d,e,f,g}	1	e
∞	8	.	2	.	9	∞	{a,b,d,f,g}	2	d
∞	8	.	.	.	9	6	{a,b,f,g}	3	g
∞	8	.	.	.	7	.	{a,b,f}	4	f
10	8	.	.	.	.	.	{a,b}	5	b
10	.	.	.	.	.	.	{a}	6	a
.	.	.	.	.	.	.	$\emptyset$ STOP		

$d(c,\cdot)$ = 10 8 0 2 1 7 6



2(a)

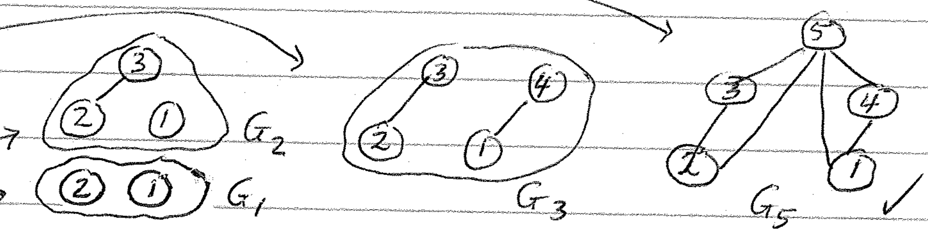
4, 2, 2, 2, 2

1, 1, 1, 1

0, 1, 1

1, 1, 0

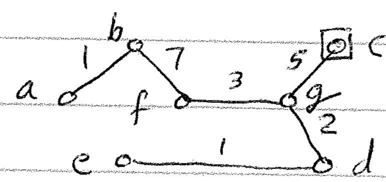
0 0



(b)

a	b	c	d	e	f	g	E(T)	V(T)	i	x_0
∞	∞	0	∞	∞	∞	∞	$\emptyset$	{c}	0	c
∞	8	.	8	∞	∞	5	{cg}	{c,g}	1	g
∞	8	.	2	∞	3	.	{cg,gd}	{c,g,d}	2	d
∞	8	.	.	1	3	.	{cg,gd,de}	{c,g,d,e}	3	e
9	8	.	.	.	3	.	{cg,gd,de,gf}	{c,g,d,e,f}	4	f
9	7	.	.	.	.	.	{cg,gd,de,gf,fb}	{c,g,d,e,f,b}	5	b
1	.	.	.	.	.	.	{cg,gd,de,gf,fb,ba}	{c,g,d,e,f,b,a}	6	a

min. spanning tree =

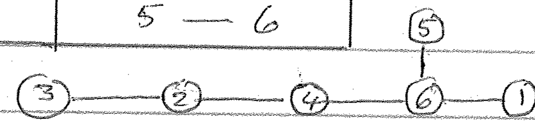


$w(T) = 5+2+1+3+7+1$
$= 19.$

3(a) $|\underset{\sim}{s}| = |\langle 6,2,4,6\rangle| = 4 \Rightarrow p = |\underset{\sim}{s}|+2 = 6$

$d_i(1)$	$d_i(2)$	$d_i(3)$	$d_i(4)$	$d_i(5)$	$d_i(6)$	$\ell(i) — s(i)$	i
$\underline{1}$	2	1	2	1	3	1 — 6	1
0	2	$\underline{1}$	2	1	2	3 — 2	2
0	1	0	2	1	2	2 — 4	3
0	0	0	1	1	2	4 — 6	4
0	0	0	0	1	1	5 — 6	

Tree corresp. to $\langle 6,2,4,6\rangle$ is:



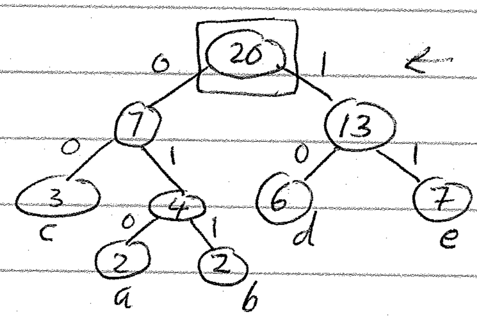
(b) $\underbrace{2, 2}, 3, 6, 7$

$\underbrace{3, 4}, 6, 7$

$\underbrace{6, 7}, 7$

$\underbrace{7, 13}$

20



← coding tree

Char.	a	b	c	d	e
Freq.	2	2	3	6	7
Codes	010	011	00	10	11
$f(c_i)\cdot\ell(c_i)$	2(3)	2(3)	3(2)	6(2)	7(2)

W.P.L. (coding)
$= 2(3) + 2(3) + 3(2) + 6(2) + 7(2)$
$= 6 + 6 + 6 + 12 + 14 = 44.$

4(a) The adjacency matrix A_G of G is defined by $A_G[i,j]$ = no. of edges from i to j in G.

(b) We will prove the result by induction on n. For $n=1$, we have $A_G^1[i,j]$ = no. of edges from i to j in G = no. of walks of length 1 from i to j in G. So the result is true for $n=1$.

Now suppose the result is true for walks of length for all i & j.
Then no. of walks of length n from i to j in G = $(A_G^n)[i,j]$ for all i & j.

So $\begin{pmatrix}\text{No. of walks of length}\\ n+1 \text{ from } i \text{ to } j \text{ in } G\end{pmatrix} = \sum_{k=1}^{p} \begin{pmatrix}\text{no. of walks of}\\ \text{length } n \text{ from } i \text{ to } k\end{pmatrix} \cdot \begin{pmatrix}\text{no. of walks of}\\ \text{length 1 from } k \text{ to } j\end{pmatrix}$

$$= \sum_{k=1}^{p} A_G^n[i,k]\,.\,A^1[k,j] = A_G^{n+1}[i,j]$$

by the definition of matrix multiplication. So if the result is true for n, it will be true for $n+1$. By the Principle of Math. Induction it follows that the result is true for all $n \in \mathbb{Z}^+$.

5(a) The distance from u to v in a weighted digraph G is defined by

$$d(u,v) = \begin{cases} \text{length of the shortest directed path from } u \text{ to } v \text{ in } G; \\ +\infty, \text{ if there is no directed path from } u \text{ to } v \text{ in } G. \end{cases}$$

(b) Let u and v be any two vertices in G^c. We have to show that there is a path from u to v in G^c. There are two cases.

Case (i): $\overline{uv} \notin E(G)$: In this case $\overline{uv}$ must be in $E(G^c)$. So $u - v$ will be a path from u to v in G^c.

Case (ii): $\overline{uv} \in E(G)$: In this case u and v will belong to the same component of G. Since G is a disconnected graph, G must have at least one more component. Let w be a vertex in one of these other components. Then $\overline{uv}$ & $\overline{wv}$ will not be in $E(G)$ – otherwise w will be in the same component as u & v. So $\overline{uw}$ & $\overline{wv}$ will be in $E(G^c)$. Thus $u - w - v$ will be a path from u to v in G^c.

Hence in both cases we found a path from u to v in G^c. Therefore G^c must be connected bec. u & v were arb.

6(a) A legal flow in N is any function $f: E(G) \to [0,\infty)$ such that (i) $f(e) \leq c(e)$ for each $e \in E(G)$ and (ii) $\sum_{e \in In(v)} f(e) = \sum_{e \in Out(v)} f(e)$ for any $v \in V(G) - \{s,t\}$.

(b) Let k = no. of vertices of degree 2 in T and l = no. of vertices of degree 1 in T. Then $5 + 10 + k + l = 50 \ldots$ (1)

Now from a Theorem in class, sum of degrees in $T = 2|E(T)|$ and from another Theorem in class $|E(T)| = p - 1 = 50 - 1 = 49$.

So $5(4) + 10(3) + k(2) + l(1) = 2(49) \ldots$ (2). Hence

$20 + 30 + 2k + l = 98$, so $2k + l = 48 \ldots$ (3)

But from equation (1) above $k + l = 35 \ldots$ (4)

Subtracting eq. (4) from eq. (3) gives us $k = 13$.

Hence T has 13 vertices of degree 2. [By the way, solving for l, gives us $l = 22$, which is the no. of leaves in T.]