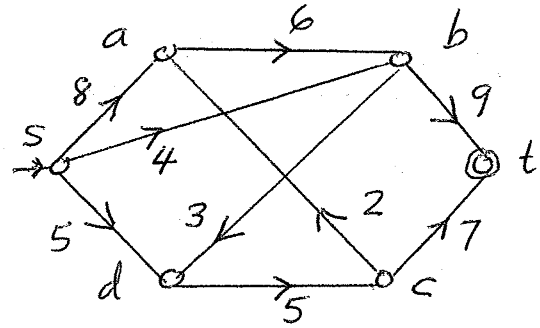
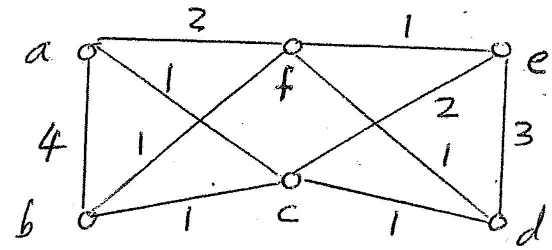
MAD 3305 - GRAPH THEORY FLORIDA INT'L UNIV.
TEST #2 - FALL 2015 TIME: 75 min.

Answer *all 6* questions. **No Calculators or Cellphones are allowed.** *An unjustified answer will receive little or no credit. BEGIN EACH OF THE 6 QUESTIONS ON A SEPARATE PAGE.*

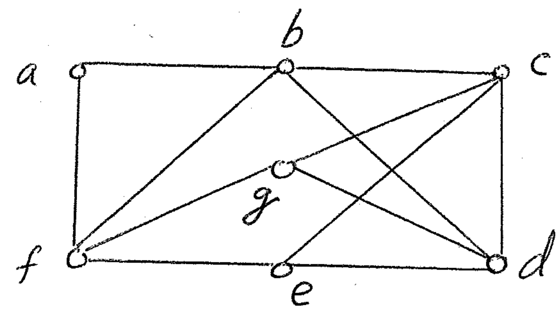
(15) 1. Find a maximal flow f^* in the network on the right by using the *Ford-Fulkerson Algorithm*. Also find the source-separating set of vertices S^* corresponding to f^*.



(15) 2. Find a *minimum postman walk* of the graph on the right by using the *Postman Algorithm*; and find *the total length* of your minimum postman walk?

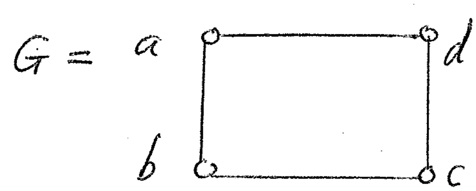


(20) 3. Determine whether or not the graph on the right is planar by using the *DMP Planarity Algorithm*. [*Show the embeddings for each step of the algorithm.*]



(20) 4(a) Find $P_G(\lambda)$ for the graph G on the right by using the *Chromatic Polynomial Algorithm.*

(b) Prove that $\chi(T) \leq 2$ for any tree T.



(15) 5(a) Define what is a *minimum Salesman walk* of a weighted graph *G*.

(b) Use *Ore's Theorem* to prove that any graph G with $\deg(x)+\deg(y)\geq p-1$, for all pairs of non-adjacent vertices x & y, has a *Hamilton path*. Here $p=|V(G)|$.

(15) 6(a) Define what is a *maximal planar* G and what is a *simple polyhedron.*

(b) Let G be a *maximal planar graph* with $p\geq 3$ vertices and E be a planar embedding of G. Prove that each region of E is bounded by exactly 3 edges.

1 (a) 1st aug. semi-path

$s \xrightarrow{(0,4)} b \xrightarrow{(0,9)} t$

Slacks = 4, 9 $\qquad \mu_1 = 4$

2nd aug. semi-path

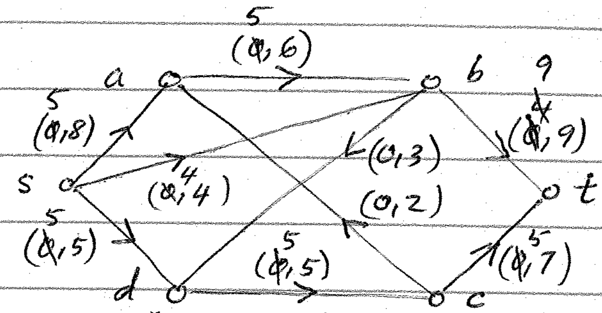
$s \xrightarrow{(0,8)} a \xrightarrow{(0,6)} b \xrightarrow{(4,9)} t$

Slacks = 8, 6, 5 $\qquad \mu_2 = 5$

3rd aug. semi-path

$s \xrightarrow{(0,5)} d \xrightarrow{(0,5)} c \xrightarrow{(0,7)} t$

Slacks = 5, 5, 7 $\qquad \mu_3 = 5$



$Val(f^*)$ = net flow into t
$= f^*(\vec{bt}) + f^*(\vec{ct}) = 9 + 5 = 14$

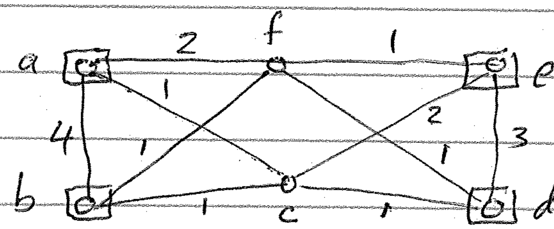
(b) $S^* = \{u \in V :$ there is an aug. semi-path from s to $u\} = \{s, a, b, d\}$

$c(S^*)$ = sum of outward capacities $= c(\vec{dc}) + c(\vec{bt}) = 5 + 9 = 14 = Val(f^*)$.

2 (a) Odd vertices are: a, b, d, e

Distances

	a	b	d	e
a	'	2	2	3
b		'	2	2
d			'	2
e				''

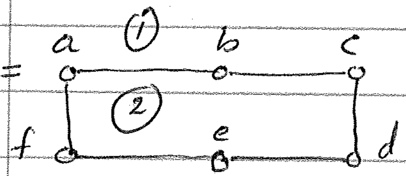


$\{a,b\} + \{d,e\}$	$\{a,d\} + \{b,e\}$	$\{a,e\} + \{b,d\}$
$2+2=4$ ✓	$2+2=4$	$2+3=5$

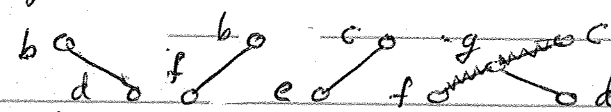
(b) A min. postman walk is: $a \overset{4}{—} b \overset{1}{—} c \overset{1}{—} b \overset{1}{—} f \overset{1}{—} e \overset{1}{—} f \overset{1}{—} d \overset{3}{—} e \overset{2}{—} c \overset{1}{—} d \overset{1}{—} f \overset{2}{—} a \overset{1}{—} c \overset{1}{—} a$

Total length = 21.

3. $H_1 =$

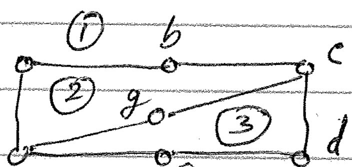


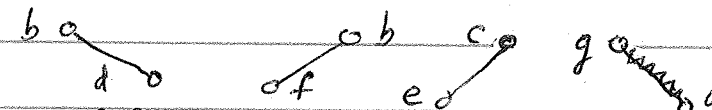
Segments of G relative to H_i



$R_1(S)$. $\{1,2\}$ $\{1,2\}$ $\{1,2\}$ $\{1,2\}$

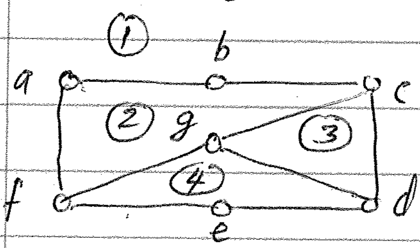
$H_2 =$

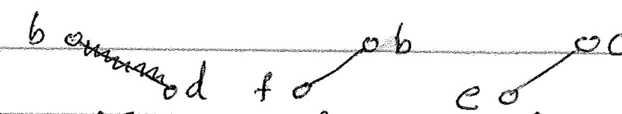




$R_2(S)$: $\{1\}$ $\{1,2\}$ $\{1,3\}$ $\{3\}$

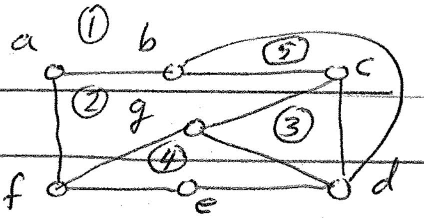
$H_3 =$

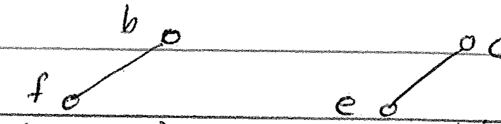




$R_3(S)$: $\{1\}$ $\{1,2\}$ $\{1\}$

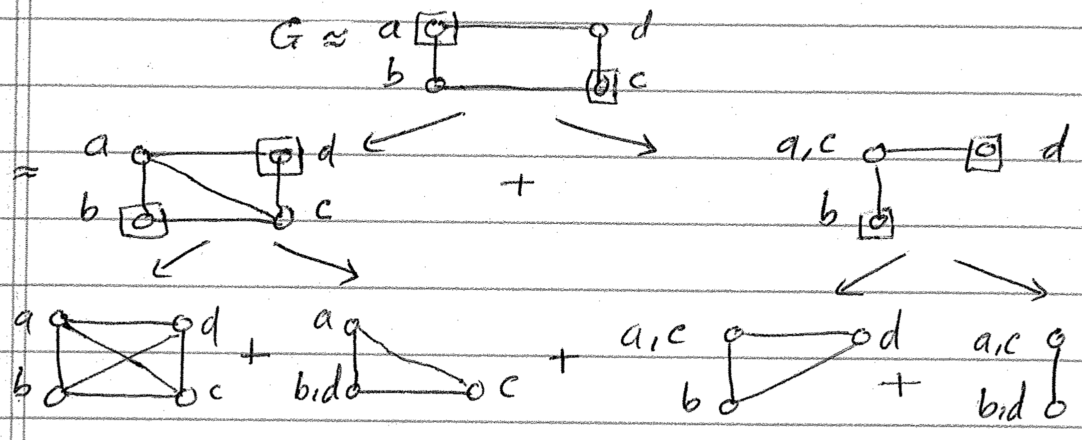
3 H_4 =





$R_4(5)$: $\{1,2\}$ $\qquad$ $\emptyset$ $\qquad$ G is NON-PLANAR

4(a)



$$P_G(\lambda) = P_{K_4}(\lambda) + 2P_{K_3}(\lambda) + P_{K_2}(\lambda) \quad \text{bec.} \quad G \approx K_4 + 2K_3 + K_2$$
$$= \lambda(\lambda-1)(\lambda-2)(\lambda-3) + 2\lambda(\lambda-1)(\lambda-2) + \lambda(\lambda-1)$$
$$= \lambda(\lambda-1)\left[(\lambda-2)(\lambda-3) + 2(\lambda-2) + 1\right] = \lambda(\lambda-1)(\lambda^2-3\lambda+3).$$

(b) Choose any vertex v_0 of T and designate it as the root to get a rooted tree $\langle T, v_0\rangle$. Now color the even levels of $\langle T, v_0\rangle$ with color #1 and the odd levels with color #2. We claim that this will be a legal coloring of G. Now suppose this was <u>not</u> a legal coloring of G. Then we can find two adjacent vertices X & Y with the same color. Since X & Y are adjacent, they must be in adjacent levels — so one of them will be in an even level and the other in an odd level. But this means that they will get different colors — a contradiction. Hence we got a legal coloring of T with 2 colors. $\therefore \chi(T) \le 2$.

5(a) A minimum salesman walk of G is a closed walk which includes every vertex of G and is of the shortest possible total length.

(b) Let $V(G) = \{v_1, \ldots, v_p\}$ and let G' be the graph defined by $V(G') = V(G) \cup \{v_{p+1}\}$ and $E(G') = E(G) \cup \{v_1v_{p+1}, v_2v_{p+1}, \ldots, v_pv_{p+1}\}$, where v_{p+1} is a new vertex.

5(b) Then $|V(G')| = p+1$ and for any pair of non-adjacent vertices x & y in G' we will have

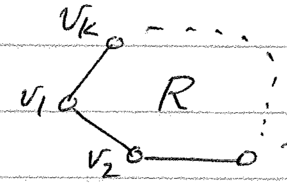
$$\begin{aligned} deg_{G'}(x) + deg_{G'}(y) &= [deg_G(x)+1] + [deg_G(y)+1] \\ &= deg_G(x) + deg_G(y) + 2 \\ &\geq (p-1) + 2 = p+1 = |V(G')|. \end{aligned}$$

So by Ore's Theorem G' will have a Hamilton cycle C. Now if we delete the vertex v_{p+1} from this cycle C, we will get a Hamilton path P in G. So any graph G with $deg(x) + deg(y) \geq p-1$ for all non-adj. vertices x & y will have a Hamilton path.

6 (a) The graph G is maximal planar if G is planar and for any pair of non-adjacent vertices x & y in G, $G \cup \{\overline{xy}\}$ is non-planar. A simple polyhedron is any solid figure bounded by plane polygonal faces that can be continuously distorted into a solid sphere.

(b) Let G be a maximal planar graph with $p \geq 3$ and E be a planar embedding of G. Now suppose there is a region R of E that is bounded by 4 or more edges. Then the boundary of the region will give us a cycle $\langle v_1, v_2, v_3, \ldots, v_k, v_1 \rangle$ where $k \geq 4$. We will show this leads to a contradiction. There are 2 cases.

Case (i): $\overline{v_1 v_3} \notin E(G)$. In this case we can add the edge $\overline{v_1 v_3}$ to G inside the region R and so contradict the fact that G is maximal planar.



Case (ii): $\overline{v_1 v_3} \in E(G)$. In this case the edge $\overline{v_1 v_3}$ in E will prevent $\overline{v_2 v_k}$ from being an edge of G because there will be no way to embedd $\overline{v_2 v_k}$ in E without crossing $\overline{v_1 v_3}$ outside the region R. Hence we can add $\overline{v_2 v_k}$ to E inside R to contradict the fact that G is maximal planar. Hence every region of E must be bounded by 3 edges (bec. G is a graph).