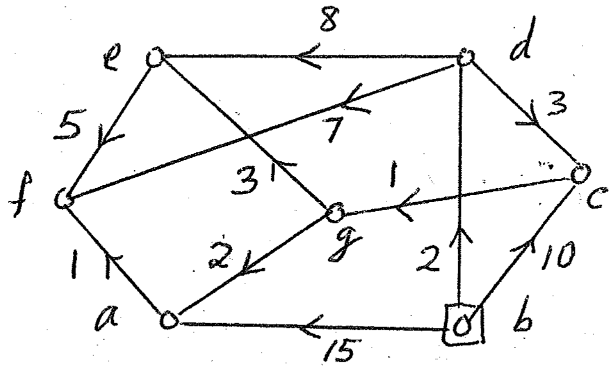
MAD 3305 - GRAPH THEORY | FLORIDA INT'L UNIV.
TEST #1 - FALL 2016 | TIME: 75 min.

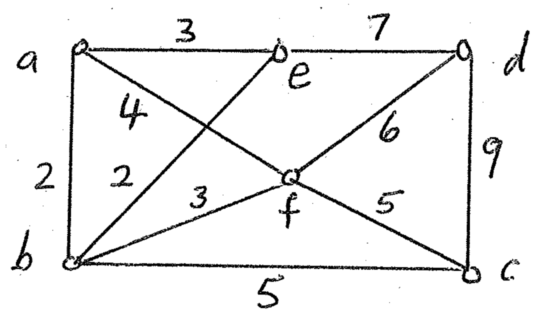
Answer *all 6* questions. ***No Calculators or Cellphones are allowed.*** *An unjustified answer will receive little or no credit. BEGIN EACH OF THE 6 QUESTIONS ON A SEPARATE PAGE.*

(15) 1. Find the *distances* from b to each of the other vertices of the graph on the right by using *Dijkstra's Algorithm.*



(20) 2(a) Find a *graph* with degree sequence $\langle 4, 3, 3, 3, 3\rangle$ by using the *Graphical Sequence Algorithm.*

(b) For the graph on the right, find a *minimal spanning tree* by using *Kruskal's Algorithm.*



(20) 3(a) Find the *tree* that corresponds to the sequence $\langle 2, 4, 2, 3\rangle$ via *Prufer's Tree Decoding Algorithm.*

(b) The five characters *a, b, c, d, e* occur with frequencies *2, 3, 5, 7, 8* respectively. Find an *optimal binary coding* for these five characters and the *weighted-path length* of your coding by using *Huffman's algorithm.*

(15) 4(a) Define what is the *adjacency matrix A* of a graph G with $V(G) = \{1,2,3,..., p\}$.

(b) Prove that: $A^n[i, j]$ = no. of walks of length n from vertex i to vertex j in G.

(15) 5(a) Define what is a *connected component H* of a disconnected graph G.

(b) Let G be a graph and G^c be the complement of G. If G^c is a disconnected graph, prove that G must be a connected graph.

(15) 6(a) Define what is a *legal flow f* in a network $N = \langle G,s,t,c\rangle$ and define $Val(f)$.

(b) A certain tree T has 10 vertices of degree 5, 20 of degree 4, 20 of degree 3, and the rest of degree 1 or 2. How many *leaves* does T have?

[You may use any theorem that was proved in class in solving 6(b).]

MAD 3305 - GRAPH THEORY   Florida Int'l Univ.

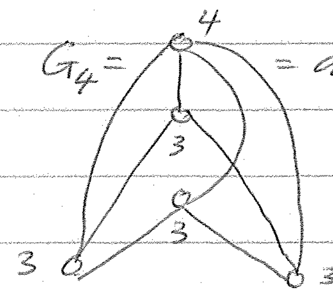
Solutions to Test #1   Fall 2016

1.

$L(a)$	$L(b)$	$L(c)$	$L(d)$	$L(e)$	$L(f)$	$L(g)$	T	i	V_0
∞	0	∞	∞	∞	∞	∞	$\{a,b,c,d,e,f,g\}$	0	b
15	,	10	$\underline{2}$	∞	∞	∞	$\{a,c,d,e,f,g\}$	1	d
15	,	$\underline{5}$	,	10	9	∞	$\{a,c,e,f,g\}$	2	c
15	,	,	,	10	9	$\underline{6}$	$\{a,e,f,g\}$	3	g
$\underline{8}$	,	,	,	9	9	,	$\{a,e,f\}$	4	a
,	,	,	,	9	$\underline{9}$	,	$\{e,f\}$	5	f
,	,	,	,	9	,	,	$\{e\}$	6	e.
,	,	,	,	,	,	,	$\emptyset$, STOP		

$d(b,\cdot) =$ 8 0 5 2 9 9 $\underline{6}$

2 (a) $4, \underline{3,3,3,3}$ $\longrightarrow$ $G_4 =$



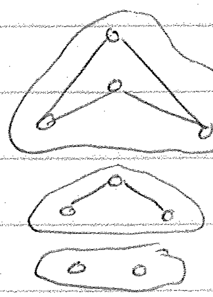
= answer

$2, \underline{2,2}\ 2$ $\longrightarrow$

$1,1,2$

$2, \underline{1,1}$ $\longrightarrow$

$\underline{0,0}$ $\longrightarrow$



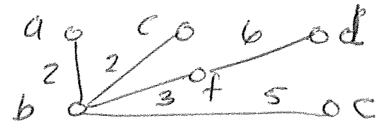
G_3, G_2, G_1

2(b) $\overline{ab}, \overline{be}, \overline{ae}, \overline{bf}, \overline{af}, \overline{bc}, \overline{cf}, \overline{df}, \overline{de}, \overline{cd}$

$e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}$

$E(T)$	Parts of the Partition	i	endpts(e_i)
$\emptyset$	$\{a\}\{b\}\{c\}\{d\}\{e\}\{f\}$	1	$\{a,b\}$
$\{\overline{ab}\}$	$\{a,b\}\{c\}\{d\}\{e\}\{f\}$	2	$\{b,e\}$
$\{\overline{ab}, \overline{be}\}$	$\{a,b,e\}\{c\}\{d\}\{f\}$	3	$\{a,e\}$
don't add $\overline{ae}$	unchanged	4	$\{b,f\}$
$\{\overline{ab}, \overline{be}, \overline{bf}\}$	$\{a,b,e,f\}\{c\}\{d\}$	5	$\{a,f\}$
don't add $\overline{af}$	unchanged	6	$\{b,c\}$
$\{\overline{ab}, \overline{be}, \overline{bf}, \overline{bc}\}$	$\{a,b,c,e,f\}\{d\}$	7	$\{c,f\}$
don't add $\overline{cf}$	unchanged	8	$\{d,f\}$
$\{\overline{ab}, \overline{be}, \overline{bf}, \overline{bc}, \overline{df}\}$	$\{a,b,c,d,e,f\}$ STOP	(bec. partition has 1 part)	

2(b) A Minimal Spanning tree =

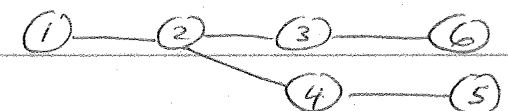


$p = |\underset{\sim}{S}| + 2 = 4 + 2 = 6$

3(a)

$d_i(1)$	$d_i(2)$	$d_i(3)$	$d_i(4)$	$d_i(5)$	$d_i(6)$	i	$l(i)$ — $s(i)$
$\underline{1}$	$\underline{3}$	2	2	1	1	1	1 — 2
0	2	2	$\underline{2}$	$\underline{1}$	1	2	5 — 4
0	$\underline{2}$	2	$\underline{1}$	0	1	3	4 — 2
0	$\underline{1}$	$\underline{2}$	0	0	1	4	2 — 3
0	0	$\underline{1}$	0	0	$\underline{1}$	5 STOP	3 — 6

T =



3(b)

a b c d e

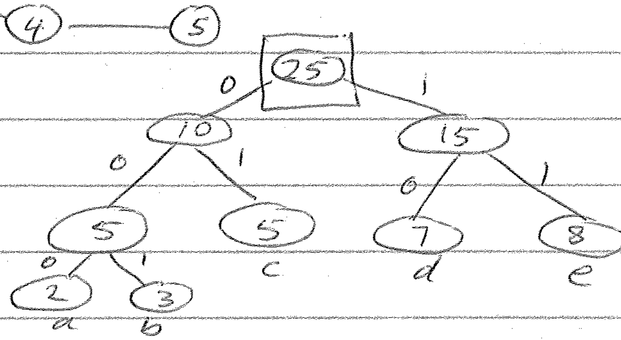
2 3 5 7 8

5, 5, 7, 8

7, 8, → 10

10, 15

25



Char	a	b	c	d	e
Code	000	001	01	10	11
$f(c_i)$	2	3	5	7	8
$l(c_i)$	3	3	2	2	2

$WPL(\text{coding}) = 2(3) + 3(3) + 5(2) + 7(2) + 8(2) = 55.$

4(a) A is the $p \times p$ matrix with $A[i,j]$ = no. of edges from i to j.

4(b) We will prove the result by induction on n. (<u>Basis</u>): When $n=1$, we have: $\begin{pmatrix}\text{no. of walks of}\\ \text{length 1 from } i \text{ to } j\end{pmatrix} = \begin{pmatrix}\text{no. of edges}\\ \text{from } i \text{ to } j\end{pmatrix} = A^1[i,j]$.

So the result is true for $n=1$. (<u>Ind. Step</u>): Now suppose the result is true for n. Then $\begin{pmatrix}\text{no. of walks of}\\ \text{length } n \text{ from } i \text{ to } j\end{pmatrix} = (A^n)[i,j]$. So

$$\begin{pmatrix}\text{no. of walks of length}\\ n+1 \text{ from } i \text{ to } j\end{pmatrix} = \sum_{k=1}^{p} \begin{pmatrix}\text{no. of walks of}\\ \text{length } n \text{ from } i \text{ to } k\end{pmatrix} \cdot \begin{pmatrix}\text{no. of walks of}\\ \text{length 1 from } j \text{ to } k\end{pmatrix}$$

$$= \sum_{k=1}^{p} (A^n)[i,k] \cdot A^1[k,j] = (A^{n+1})[i,j].$$

So if the result is true for n, it will also be true for $n+1$.

(<u>Concl.</u>) By the Princ. of Math Ind., the result is now true for all n.

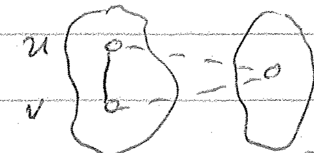
5(a) H is a connected component of G if H is a maximal connected subgraph of G. (Maximal means that there is no connected subgraph H' of G which properly contains H.)

5(b) Let u and v be any two vertices of G. We will show that there is always a path from u to v in G. There are 2 cases.

<u>Case(i)</u>: $\overline{uv} \notin E(G^c)$: In this case $\overline{uv} \in E(G)$ and $u \circ\!\!\longrightarrow\!\!\circ v$ readily provides a path from u to v in G.

<u>Case(ii)</u>: $\overline{uv} \in E(G^c)$: In this u and v will belong to the same component of G^c. Since G^c is disconnected, it must have a vertex, w say, which is in a different component from the one that contains u & v. Now $\overline{uw}$ & $\overline{wv}$ must both be in $E(G)$ otherwise w would be in the same component as u & v. Hence $u \circ\!\!\longrightarrow\!\!\circ w \circ\!\!\longrightarrow\!\!\circ v$ provides a path from u to v in G.

So in either case we get a path from u to v in G. $\therefore$ G is connected.



The situation in G^c

6(a) A legal flow in N is any function $f: E(G) \to [0, \infty)$ such that (i) $f(e) \le c(e)$ for all $e \in E(G)$ & (ii) $\sum_{e \in In(v)} f(e) = \sum_{e \in Out(v)} f(e)$ for all $v \in V(G) - \{s,t\}$

$$Val(f) = \sum_{e \in In(t)} f(e) - \sum_{e \in Out} f(e).$$

6(b) Let p = no. of vertices in T, k = no. of vertices of deg. 2, an l = no. of leaves in T. Then $p = 10+20+20+k+l = 50+k+l$. By a theorem from class, $|E(T)| = p-1$ and by another theorem sum of degrees in $T = 2|E(T)| = 2(p-1)$. So

$$5(10) + 4(20) + 3(20) + 2(k) + 1(l) = 2(50+k+l-1)$$

$\therefore$ $190 + \cancel{2k} + l = 100 + \cancel{2k} + 2l - 2$. So $190 + 2 - 100 = l$

$\therefore$ no. of leaves in $T = l = 192 - 100 = 92$.