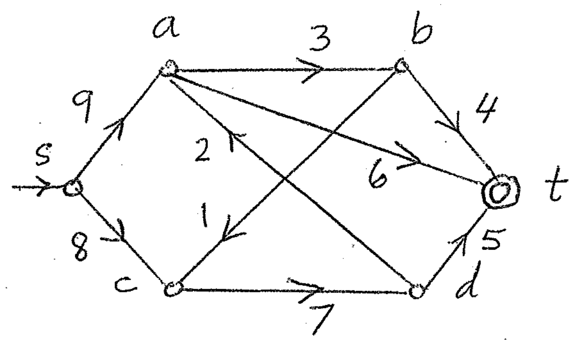
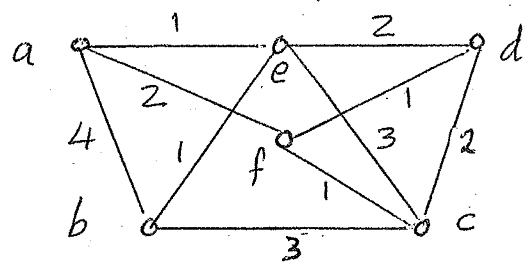
MAD 3305 - GRAPH THEORY | FLORIDA INT'L UNIV.
TEST #2 - FALL 2016 | TIME: 75 min.

Answer *all 6* questions. ***No Calculators or Cellphones are allowed.*** *An unjustified answer will receive little or no credit. BEGIN EACH OF THE 6 QUESTIONS ON A SEPARATE PAGE.*

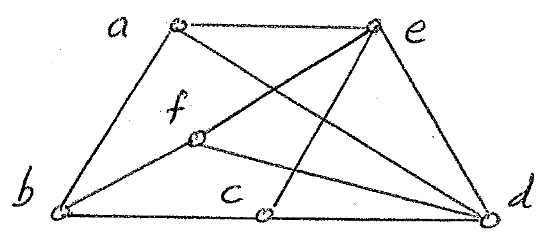
(15) 1. Find a maximal flow f^* in the network on the right by using the *Ford-Fulkerson Algorithm*. Also find the source-separating set of vertices S^* corresponding to f^*.



(15) 2. Find a *minimum postman walk* of the graph on the right by using the *Postman Algorithm*; and find *the total length* of your minimum postman walk?

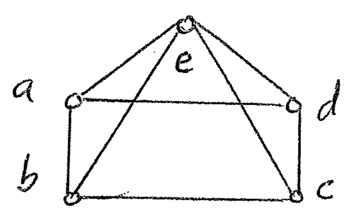


(18) 3. Determine whether or not the graph on the right is planar by using the *DMP Planarity Algorithm*. [*Show the embeddings for each step of the algorithm.*]



(22) 4(a) Find $P_G(\lambda)$ for the graph G on the right by using the *Chromatic Polynomial Algorithm.*

(b) Prove $P_G(\lambda) = P_{G\cup\{ab\}}(\lambda) + P_{G\circ\{ab\}}(\lambda)$ if a &b are non-adjacent vertices in G.



(15) 5(a) Let G_0 be a connected multi-graph. Define what is an *open Euler-trail* of G_0.

(b) Prove G_0 has an open Euler-trail $\Leftrightarrow$ G_0 has exactly 2 vertices of odd degree. [*You may use the Euler-Circuit theorem in question #5, if needed.*]

(15) 6(a) Define what is the *dual of* the planar graph G, w.r.t. the planar embedding $\mathcal{E}$.

(b) Let $\mathcal{E}$ be a planar embedding of G in which no region is bounded by less than 6 edges. If G has p vertices, q edges & G is not a forest; prove $q \leq (3p-6)/2$. [*You may use any theorem that was proved in class for Qu. #6, if needed.*]

MAD 3305 – Graph Theory
Solutions to Test #2

Florida Internat. Univ.
Fall 2016

1. 1st aug. semi-path

$s \xrightarrow{(0,9)} a \xrightarrow{(0,3)} b \xrightarrow{(0,4)} t$

slacks: 9, 3, 4; $\mu_1 = 3$

2nd aug. semi-path

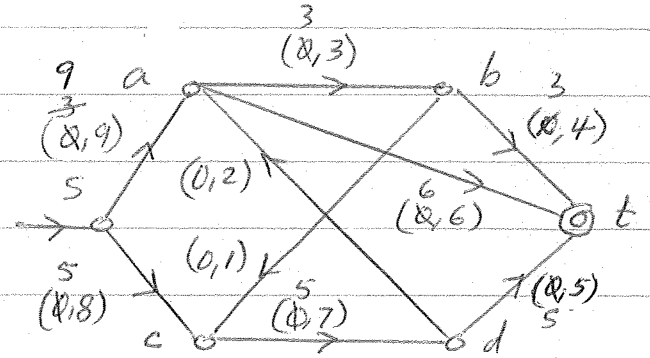
$s \xrightarrow{(0,8)} c \xrightarrow{(0,7)} d \xrightarrow{(0,5)} t$

slacks: 8, 7, 5; $\mu_2 = 5$

3rd aug. semi-path

$s \xrightarrow{(3,9)} a \xrightarrow{(0,6)} t$

slacks: 6, 6; $\mu_2 = 6$



$S^* = \{u \in V(G) :$ there is an aug. semi-path from s to $u\} = \{s, a, c, d\}$

$c(S^*) = c(\vec{ab}) + c(\vec{at}) + c(\vec{dt}) = 3+6+5 = 14$

$Val(f^*) = f^*(\vec{bt}) + f^*(\vec{at}) + f^*(\vec{dt}) = 3+6+5 = 14 = c(S^*)$, so we're done.

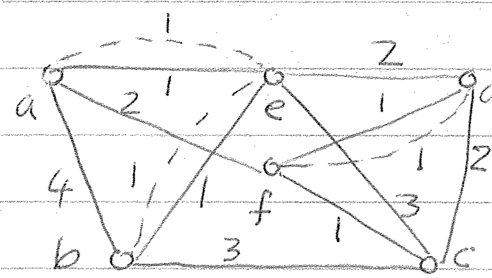
2.

	a	b	d	f
a	·	2	3	2
b	·	·	3	4
d	·	·	·	1
f	·	·	·	·

$\{a,b\} + \{d,f\} = 2+1 = 3$

$\{a,d\} + \{b,f\} = 3+4 = 7$

$\{a,f\} + \{b,d\} = 2+3 = 5$

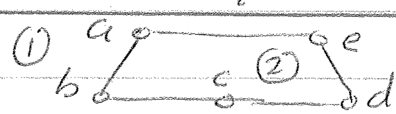


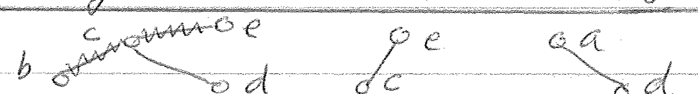
Postman walk $= f \overset{1}{-} d \overset{1}{-} f \overset{1}{-} c \overset{2}{-} d \overset{2}{-} e \overset{3}{-} c \overset{3}{-} b \overset{1}{-} e \overset{1}{-} b \overset{4}{-} a \overset{1}{-} e \overset{1}{-} a \overset{2}{-} f$

Length of postman walk $= 1+1+1+2+2+3+3+1+1+4+1+1+2 = \boxed{23}$.

3. i | H_i | Segments of G relative to H_i

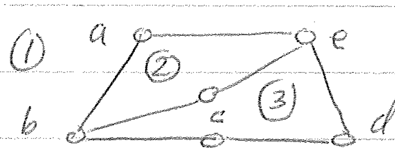
1





R_i : {1,2} {1,2} {1,2}

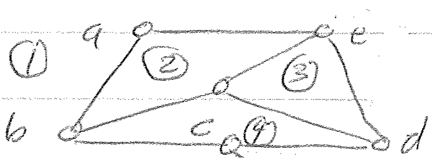
2.



~~f — d~~ c — e a — d

R_i : {3} {1,3} {1}

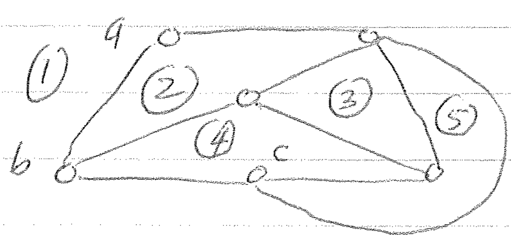
3.



~~c — e~~ a — d

R_i : {1} {1}

4

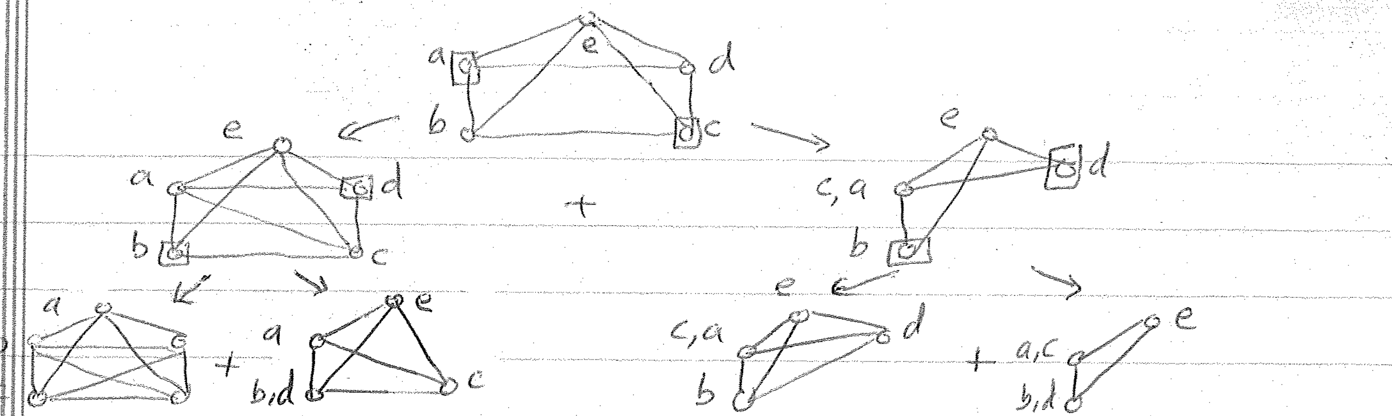


a — d

R_i : $\emptyset$

$\therefore$ G is NON-PLANAR.

4(a)



$P_G(\lambda) = P_{K_5}(\lambda) + 2P_{K_4}(\lambda) + P_{K_3}(\lambda) = \lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4) + 2\lambda(\lambda-1)(\lambda-2)(\lambda-3)$
$+ \lambda(\lambda-1)(\lambda-2) = \lambda(\lambda-1)(\lambda-2)[(\lambda-3)(\lambda-4)+2(\lambda-3)+1] = \lambda(\lambda-1)(\lambda-2)[\lambda^2-5\lambda+7]$.

(b) $P_G(\lambda)$ = No. of ways of legally coloring G with λ colors available
= No. of ways of coloring G with a & b getting different colors
+ No. of ways of coloring G with a & b getting the same color
$= P_{G\cup\{ab\}}(\lambda) + P_{G\circ\{ab\}}(\lambda)$

5(a) An open Euler trail of G_0 is any walk in G_0 in which each edge is used exactly once and the starting vertex $\neq$ the ending vertex.

(b) ($\Rightarrow$) Supp. G_0 has an open Euler trail, $\langle v_0, e_1, v_1, \ldots, v_{q-1}, e_q, v_q\rangle$. Then $v_0 \neq v_q$ & by adding $\overline{v_q v_0}$ to G_0 we will get a multi-graph G_1 with an Euler circuit. So by the Euler-circuit Thm, each vertex of G_1 must be of even degree. Hence G_0 has exactly two odd vertices.

($\Leftarrow$) Supp. G_0 has exactly 2 odd vertices — say u & v. The by adding an extra edge $\overline{uv}$ to G_0 we will get a graph G_1 in which all vertices are of even degree. So G_1 will have an Euler circuit Q by the Euler-circuit Thm. Now if we remove this new edge $\overline{uv}$ from Q, we will get an open Euler trail of G_0.

6(a) $V(G^*_\mathcal{E})$ = set of regions into which $\mathcal{E}$ partitions the plane & each time we have an edge between two regions of $V(G^*_\mathcal{E})$, we will get an edge e in $E(G^*_\mathcal{E})$.

(b) Since G is not a forest, G must have at least one cycle. Let $A_1, A_2, \ldots, A_r$ be the regions into which $\mathcal{E}$ partitions the plane. Then each region A_i will be bounded by at least 6 edges. (If G had no cycle, then the single region could be bounded by less than 6 edges). So $6r \leq e(A_1) + \cdots + e(A_r) \leq 2q$ because an edge is shared by at most 2 regions. $\therefore 3r \leq q$. But $q+2-p \leq q+k+1-p = r$ (by the Gen. Euler formula), so $3(q+2-p) \leq 3r \leq q$. $\therefore 2q \leq 3p-6$. Hence $q \leq (3p-6)/2$ and we are done. END.