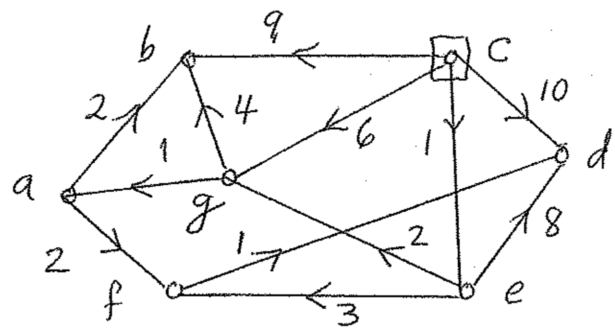
MAD 3305 - GRAPH THEORY
TEST #1 - FALL 2017

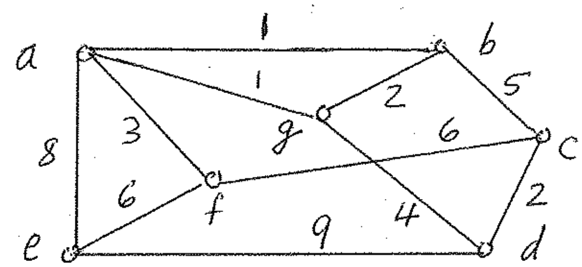
FLORIDA INT'L UNIV.
TIME: 75 min.

Answer *all 6* questions. ***No Calculators or Cellphones are allowed.*** *An unjustified answer will receive little or no credit. BEGIN EACH OF THE 6 QUESTIONS ON A SEPARATE PAGE.*

(15) 1. Find the *distances* from c to each of the other vertices of the graph on the right by using *Dijkstra's Algorithm*.



(20) 2(a) Find a *graph* with degree sequence $\langle 4, 2, 2, 2, 2, 2 \rangle$ by using the *Graphical Sequence Algorithm*.
(b) For the graph on the right, find a *minimal spanning tree* by using *Kruskal's Algorithm*.



(20) 3(a) Find the *tree* that corresponds to the sequence $\langle 5, 2, 1, 5 \rangle$ via *Prufer's Tree Decoding Algorithm*.
(b) The five characters *a, b, c, d, e* occur with frequencies *26, 10, 7, 5, 12;* respectively. Find an *optimal binary coding* for these five characters & the *weighted-path length* of your coding by using *Huffman's algorithm*.

(15) 4(a) Define what is the *adjacency matrix* of a *digraph* G with $V(G) = \{1,2,3,\ldots,p\}$.
(b) Prove that $A^n[i,j]$ = number of *directed walks* of length n from i to j in G.

(15) 5(a) Define precisely what is a *connected component* of a *graph G*.
(b) Let G be a graph with at least two connected components. Prove that G^c must have exactly one component.

(15) 6(a) Define what is a *legal flow f* in a network $N = \langle G,s,t,c \rangle$ and define what is the *value* of the flow *f*.
(b) A certain tree T has 2 vertices of degree 5, 2 of degree 4, 5 of degree 3, and the rest of degree 1 or 2. If the number of *leaves* is half the number of vertices of degree 2, how many *vertices* does T have?
[*You may use any theorem that was proved in class to answer question #6.*]

MAD 3305 – Graph Theory Florida International Univ.

Solutions to Test #1 Fall 2017

1.

	L(a)	L(b)	L(c)	L(d)	L(e)	L(f)	L(g)	T	i	v_0
	∞	∞	<u>0</u>	∞	∞	∞	∞	{a,b,c,d,e,f,g}	0	c → b, d, e, g
	∞	9	.	10	<u>1</u>	∞	6	{a,b,d,e,f,g}	1	e → d, f, g
	∞	9	.	9	.	4	<u>3</u>	{a,b,d,f,g}	2	g → a, b
	<u>4</u>	7	.	9	.	4	.	{a,b,d,f}	3	a → b, f
	.	6	.	9	.	<u>4</u>	.	{b,d,f}	4	f → d
	.	6	.	<u>5</u>	.	.	.	{b,d}	5	d → nothing
	.	<u>6</u>	.	.	.	.	.	{b}	6	b → nothing
	.	.	.	.	.	.	.	∅ STOP		
d(c,·)=	4	6	0	5	1	4	3			

2(a) 4, <u>2, 2, 2, 2, 2</u>

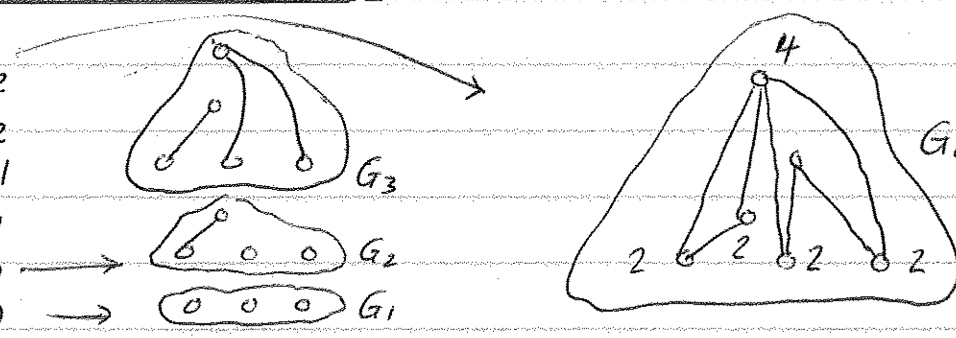
1, 1, 1, 1, 2

2, <u>1, 1</u>, 1, 1

0, 0, 1, 1

1, <u>1</u>, 0, 0 →

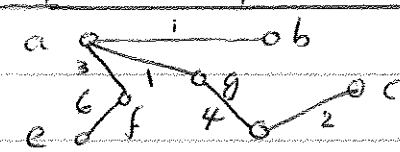
0, 0, 0 →



(b)

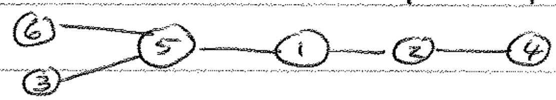
	E(T)	Parts of partition	i	endpoint(e_i)
$\overline{ab} = e_1 = 1$	∅	{a}{b}{c}{d}{e}{f}{g}	1	{a, b}
$\overline{ag} = e_2 = 1$	{$\overline{ab}$}	{a,b}{c}{d}{e}{f}{g}	2	{a, g}
$\overline{bg} = e_3 = 2$	....∪{$\overline{ag}$}	{a,b,g}{c}{d}{e}{f}	3	{b, g}
$\overline{cd} = e_4 = 2$	don't add $\overline{bg}$	same as above	4	{c, d}
$\overline{af} = e_5 = 3$	....∪{$\overline{cd}$}	{a,b,g}{c,d}{e}{f}	5	{a, f}
$\overline{dg} = e_6 = 4$	....∪{$\overline{af}$}	{a,b,f,g}{c,d}{e}	6	{a, g}
$\overline{bc} = e_7 = 5$	....∪{$\overline{dg}$}	{a,b,c,d,f,g}{e}	7	{b, c}
$\overline{cf} = e_8 = 6$	don't add $\overline{bc}$	same as above	8	{c, f}
$\overline{ef} = e_9 = 6$	don't add $\overline{cf}$	same as above	9	{e, f}
$\overline{ae} = e_{10} = 8$	....∪{$\overline{ef}$}	{a,b,c,d,e,f,g}		
$\overline{de} = e_{11} = 9$	STOP			

min. weight spanning tree =



3(a) $|\underset{\sim}{s}| = 4 = p-2 \Rightarrow p = 6.$ $\quad d_1(x) = 1 +$ no. of times x appears in $\underset{\sim}{s}$

i	$d_i(1)$	$d_i(2)$	$d_i(3)$	$d_i(4)$	$d_i(5)$	$d_i(6)$	$l(i)$	$s(i)$
1	2	2	1 ↓	1	3 ↓	1	3	5
2	2	2 ↓	0	1 ↓	2	1	4	2
3	2 ↓	1 ↓	0	0	2	1	2	1
4	1 ↓	0	0	0	2 ↓	1	1	5
	0	0	0	0	1	1	plus 5	6

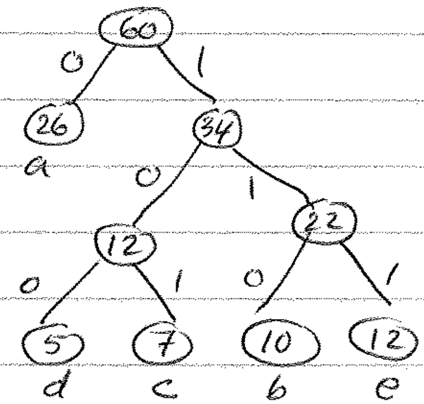
So $T =$ 

(b)

Char.	a	b	c	d	e
Freq.	26	10	7	5	12
Code	0	110	101	100	111
length	1	3	3	3	3
WPL contrib.	26 +	30 +	21 +	15 +	36

Total = 128 (21 + 15 = 36)

d c b e a
5, 7, 10, 12, 26
10, 12, 12, 26
12, 22, 26
26, 34
60



4(a) The adjacency matrix A of G is defined by $A[i,j] =$ no. of directed edges from vertex i to vertex j.

(b) We will prove the result by induction on n.

Basis: If $n=1$, then no. of directed walks of length 1 from i to j = no. of directed edges from i to j $= A^1[i,j]$. So result is true for $n=1$.

Ind. step: Suppose the result is true for n. Then no. of dir. walks ~~from~~ of length n from i to j $= A^n[i,j]$ for any i and any j. So

$$\begin{pmatrix}\text{No. of dir. walks of}\\ \text{length } n+1 \text{ from } i \text{ to } j \text{ in } G\end{pmatrix} = \sum_{k=1}^{p} \begin{pmatrix}\text{no. of dir. walks}\\ \text{of length } n\\ \text{from } i \text{ to } k \text{ in } G\end{pmatrix} \cdot \begin{pmatrix}\text{no. of dir. walks}\\ \text{of length 1 from}\\ k \text{ to } j \text{ in } G\end{pmatrix}$$

$$= \sum_{k=1}^{p} (A^n)[i,k] \cdot A^1[k,j] = (A^{n+1})[i,j]$$

by the definition of matrix multiplication. So, if the result is true for n, it will be true for $n+1$.

Conclusion. By the Principle of Mathematical Induction, it follows that the result is true for all n.

5(a) A connected component of G is a maximal-connected subgraph H of G. H is maximal-connected if there is no connected subgraph H' of G which properly contains H.

(b) We are just told that G is a disconnected graph and we have to show that G^c is connected. Let u & v be any two vertices in G^c. There are 2 cases.

Case (i): $\overline{uv} \notin E(G)$. In this case $\overline{uv} \in E(G^c)$ and we instantly get a path $u - v$ from u to v in G^c.

Case (ii) $\overline{uv} \in E(G)$. In this case u & v will be in the same component H of G. Choose a vertex w in a component that is different from H. Then $\overline{uw} \notin E(G)$ & $\overline{wv} \notin E(G)$. So $u - w - v$ will be a path from u to v in G^c, bec. $\overline{uw} \in E(G^c)$ & $\overline{wv} \in E(G^c)$.

So in either case we got a path from u to v in G^c. Hence G^c is connected and so has only one component.

6(a) A legal flow in N is any function $f: E(G) \to [0, \infty)$ such that $f(e) \le c(e)$ for each $e \in E(G)$ & $\sum_{e \in In(v)} f(e) = \sum_{e \in Out(v)} f(e)$ for each $v \in V(G) - \{s, t\}$.

Also $Val(f) = \sum_{e \in In(t)} f(e) - \sum_{e \in Out(t)} f(e)$.

(b) Let $p = |V(T)|$, k = no. of vertices of degree 2, and l = no. of leaves in T. Then $k = 2l$. So $p = 2+2+5+k+l = 9+3l$

Now we know from theorems in class that sum of degrees in $T = 2|E(T)|$ & that $|E(T)| = |V(T)| - 1 = p-1$.

So $2(5) + 2(4) + 5(3) + 2(2l) + 1(l) = 2(p-1)$

$\therefore$ $10 + 8 + 15 + 5l = 2(9 + 3l - 1)$ because $p = 9 + 3l$.

$\therefore$ $33 + 5l = 18 + 6l - 2 \implies 33 + 2 - 18 = 6l - 5l$.

$\therefore$ $l = 17$. Hence $p = 9 + 3l = 9 + 3(17) = \boxed{60}$. END