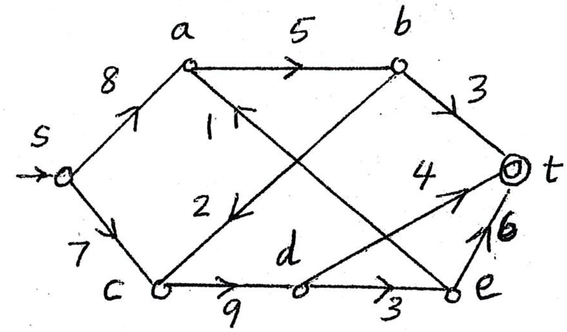
MAD 3305 - GRAPH THEORY
TEST #2 - FALL 2017

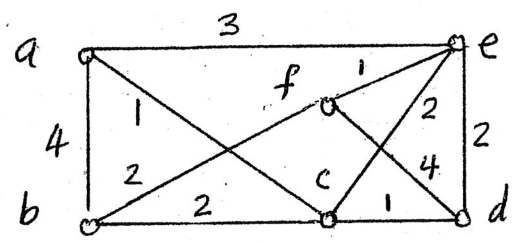
FLORIDA INT'L UNIV.
TIME: 75 min.

Answer *all* 6 questions. *No* **notes, calculators**, *or* **cellphones** *are allowed. An unjustified answer Will receive little or no credit. BEGIN EACH OF THE 6 QUESTIONS ON A SEPARATE PAGE.*

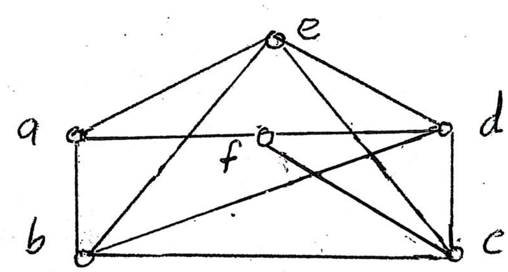
(15) 1. Find a maximal flow f^* in the network on the right by using the *Ford-Fulkerson Algorithm*. Also find the source-separating set of vertices S^* corresponding to f^*.



(15) 2. Find a *minimum postman-walk* of the graph on the right by using the *Postman Algorithm*; and find *the total length* of your minimum postman walk?

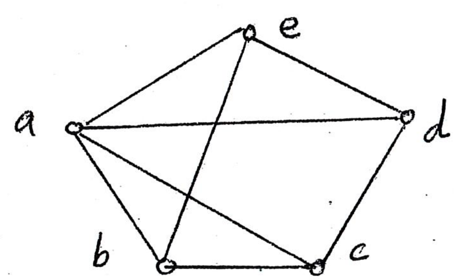


(18) 3. Determine whether or not the graph on the right is planar by using the *DMP Planarity Algorithm*. [*Show the embeddings for each step of the algorithm.*]



(22) 4(a) Find $P_G(\lambda)$ for the graph *G* on the right by using the *Chromatic Polynomial Algorithm.*

(b) If T is a non-trivial tree, prove that $\chi(T)=2$.



(15) 5(a) Define exactly when a graph G is *Hamilton-connected.*

(b) Use *Ore's theorem* to prove that any graph G with $\deg(x)+\deg(y) \geq p-1$, for all pairs of non-adjacent vertices x & y, has a *Hamilton path*. Here $p = |V(G)|$.

(15) 6(a) Define what is the *dual of* the planar graph G, w.r.t. the planar embedding $\mathcal{E}$ of G.

(b) Let $\mathcal{E}$ be a planar embedding of G in which no region is bounded by less than 10 edges. If G has p vertices and q edges, prove that $4q \leq 5p - 10$.

[*You may use any theorem that was proved in class for Qu. #6, if needed.*]

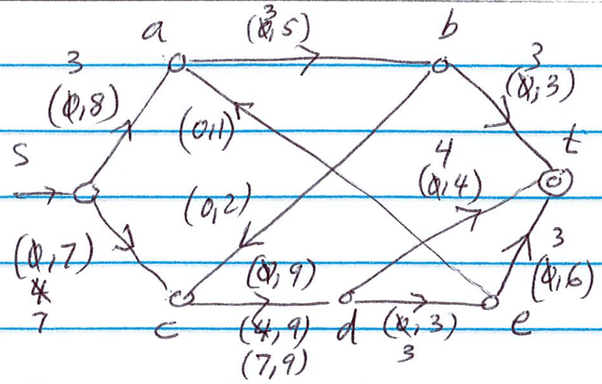
MAD 3305 - Graph Theory — Florida Int'l Univ.

Solutions to Test #2 — Fall 2017

#1. 1st aug. semi-path

$s \xrightarrow{(0,8)} a \xrightarrow{(0,5)} b \xrightarrow{(0,3)} t$, $\boxed{\mu_1 = 3}$

Slacks: 8 5 3



2nd aug. semi-path

$s \xrightarrow{(0,7)} c \xrightarrow{(0,9)} d \xrightarrow{(0,4)} t$ $\boxed{\mu_2 = 4}$

Slacks: 7 9 4

3rd aug. semi-path

$s \xrightarrow{(4,7)} c \xrightarrow{(4,9)} d \xrightarrow{(0,3)} e \xrightarrow{(0,6)} t$

Slacks 3 5 3 6 $\boxed{\mu_3 = 3}$

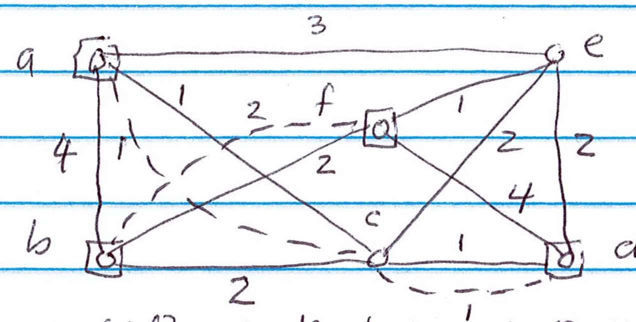
$S^* = \{s, a, b, c, d\}$

$c(S^*) = c(\vec{bt}) + c(\vec{dt}) + c(\vec{de}) = 10$

$Val(f^*) = \text{net flow into } t = 3 + 4 + 3 = 10 = c(S^*)$ ✓

#2.

dist.	a	b	d	f
a	·	3	2	4
b		·	3	2
d			·	3
f				·



$\{a,b\} + \{d,f\}$ $3+3=6$ $\{a,d\} + \{b,f\}$ $2+2=4$ ✓ $\{a,f\} + \{b,d\}$ $4+3=7$

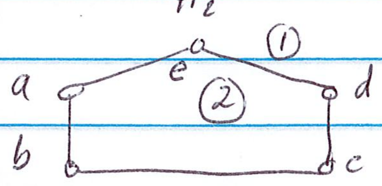
Minimum postman-walk

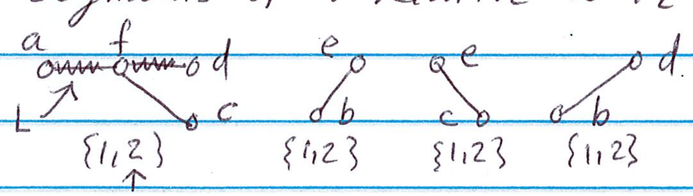
$= a \stackrel{1}{—} c \stackrel{2}{—} e \stackrel{2}{—} d \stackrel{1}{—} c \stackrel{1}{—} d \stackrel{4}{—} f \stackrel{1}{—} e \stackrel{3}{—} a \stackrel{4}{—} b \stackrel{2}{—} f \stackrel{2}{—} b \stackrel{2}{—} c \stackrel{1}{—} a$

Total length of min. postman walk = 26.

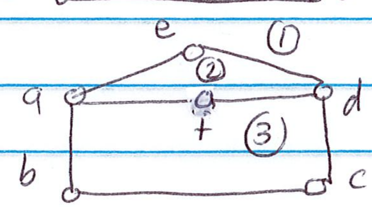
#3. i H_i Segments of G relative to H_i

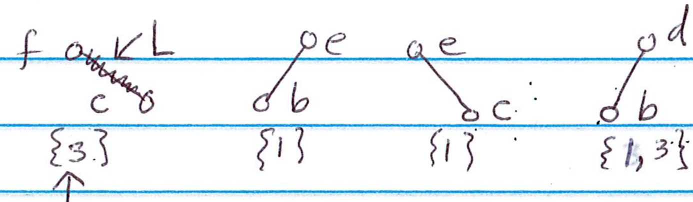
1



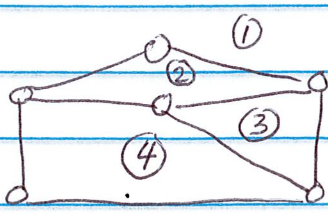


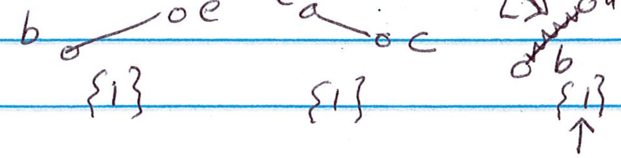
2





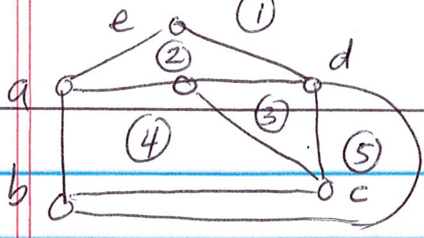
3

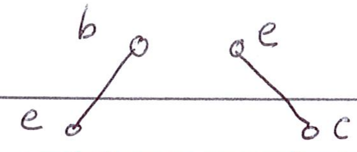




i

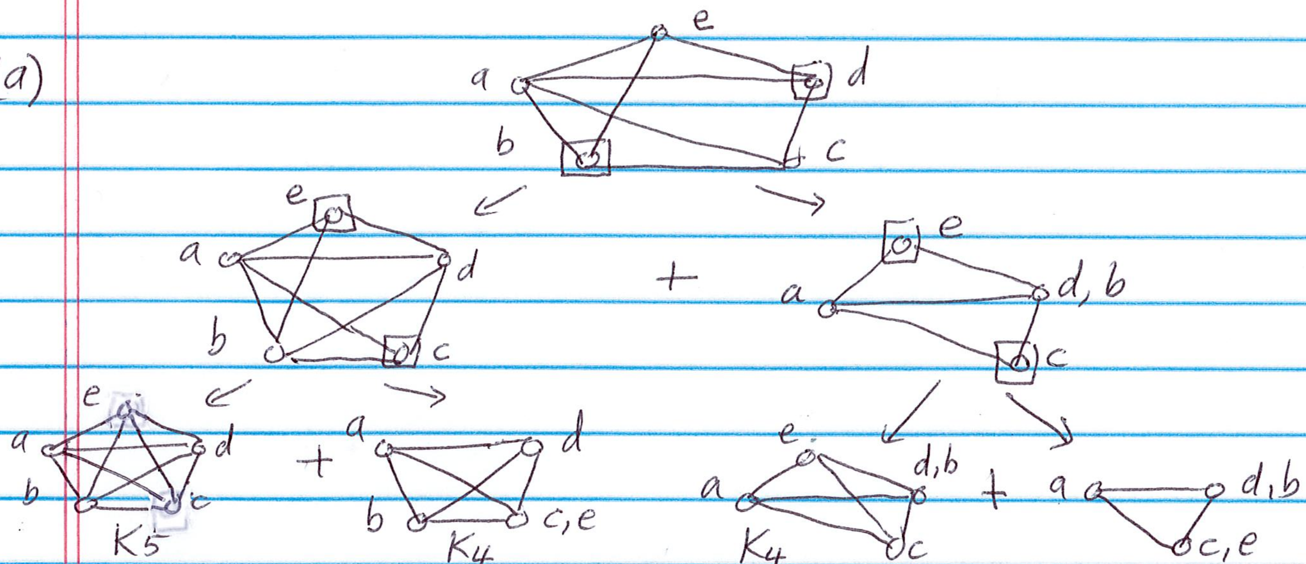
#3. 4





$\{1\}$ $\quad$ $\emptyset$ $\quad$ NON-PLANAR

#4 (a)



$$P_G(\lambda) = P_{K_5}(\lambda) + 2P_{K_4}(\lambda) + P_{K_3}(\lambda)$$
$$= \lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4) + 2\lambda(\lambda-1)(\lambda-2)(\lambda-3) + \lambda(\lambda-1)(\lambda-2)$$
$$= \lambda(\lambda-1)(\lambda-2)\left[(\lambda-3)(\lambda-4) + 2(\lambda-3) + 1\right] = \lambda(\lambda-1)(\lambda-2)\left[\lambda^2-5\lambda+7\right].$$

(b) Since a non-trivial tree has at least one edge, $\chi(T) \geq 2$ bec we need 2 colors to just color the endpoints of that edge. Now choose any vertex v in T & designate it as the root to get a rooted tree (T, v). Color the even levels of (T, v) with color #1 and the odd levels with color #2. Then this will give us a legal coloring of T. Indeed, no two vertices in the same level can be adjacent, otherwise we would get a cycle in T – which is impossible. Also no two vertices at different even levels can be adjacent – and no two vertices at different odd levels can be adjacent as well. So there is no chance for any two vertices with the same color to be adjacent. Hence this will be a legal coloring of G. $\therefore$ $\chi(G) \leq 2$. Since $\chi(G) \geq 2$, it follows that $\chi(G) = 2$ for any non-trivial tree.

5(a) A graph G is Hamilton-connected if there is a Hamilton path between any two distinct vertices of G.

(b) Let H be the graph obtained by adding a new vertex v_{p+1} to G and edges from v_{p+1} to each of the vertices of G. Then for all pairs of non-adj. vertices x & y in H

$$\deg_H(x) + \deg_H(y) = \deg_G(x)+1 + \deg_G(y)+1 \geq (p-1)+2 = p+1.$$

Since H has $p+1$ vertices, it follows from Ore's Theorem that H has a Hamilton cycle, C. Now if we remove the vertex v_{p+1} from C, we will get a Hamilton path of G.

6(a) The dual $G^*_\mathcal{E}$ of G is the graph $\langle V(G^*_\mathcal{E}), E(G^*_\mathcal{E})\rangle$ where $V(G^*_\mathcal{E})$ = set of regions into which $\mathcal{E}$ partitions the plane and for each edge between the regions R_1 & R_2 in $\mathcal{E}$, we get an edge between R_1 & R_2 in $E(G^*_\mathcal{E})$.

(b) Let $A_1, A_2, \ldots, A_r$ be the regions into which $\mathcal{E}$ partitions the plane. Since each A_i is bounded by at least 10 edges,

$$10 \cdot r \leq e(A_1) + e(A_2) + \cdots + e(A_r)$$
$$\leq 2q$$

because an edge can be counted as bounding at most 2 edges. So $5r \leq q$. But $r = q+k+1-p$ where k = no. of connected components of G. So

$$5(q+2-p) \leq 5(q+k+1-p) \quad \text{bec. } k \geq 1$$
$$= 5r \leq q$$

Hence $5q+10-5p \leq q$. So $4q \leq 5p-10$. END