Chapter 1 - 8 Exercises

GRAPH THEORY

by

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Exercises for Ch. 1

1. Determine as many isomorphisms as you can between the graphs G_1 and G_2 of Figure 1.1.3.
2. Define the complement of G using set differences.
3. Represent $K_{p_1, p_2, \ldots, p_n}$ as the join of graphs.
4. Prove that $G_1 \times G_2$ is isomorphic to $G_2 \times G_1$.
5. Determine a result analogous to Theorem 1.3.1 for digraphs.
6. Give examples to show that there are walks that are not trails and trails that are not paths.
7. Given a (p_1, q_1) graph G_1 and a (p_2, q_2) graph G_2, determine formulas for the order and size of $\overline{G_1}$, $G_1 \cup G_2$, $G_1 \times G_2$ and $G_1[G_2]$.
8. Prove or disprove: The graph $G_1[G_2]$ is isomorphic to $G_2[G_1]$.
9. Prove that if two graphs are isomorphic, then they have the same order and size and degree sequence.
10. Find all nonisomorphic graphs of order 4.
11. Show that two graphs G and H are isomorphic if, and only if, there are two bijections (1-1 and onto functions) $f_1 : V(G) \to V(H)$ and $f_2 : E(G) \to E(H)$ such that $e = uv \in E(G)$ if, and only if, $f_2(e) = f_1(u) f_2(v)$.
12. Prove that a (p, q) graph G is a complete graph if, and only if, $q = \binom{p}{2}$.

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13. Determine the order and size of $K_{p_1, p_2, \ldots, p_n}$, $n \geq 2$.
14. A (p, q) graph G is *self complementary* if G is isomorphic to $\overline{G}$. Show that if G is self complementary, then $p \equiv 0, 1 \pmod 4$.
15. Suppose $\Delta(G) = k$. Prove that there exists a supergraph H of G (that is, a graph H that contains G as a subgraph) such that G is an induced subgraph of H and H is k-regular.
16. Prove that if G is a regular bipartite graph with partite sets V_1 and V_2, then $|V_1| = |V_2|$.
17. Determine all nonisomorphic digraphs of order 4.
18. Characterize the matrices that are adjacency matrices of digraphs, that is, those matrices $A(D) = [a_{ij}]$ where $a_{ij} = 1$ if $v_i \to v_j \in E(D)$ and $a_{ij} = 0$ otherwise.
19. Determine which of the following sequences is graphical, and for those that are graphical, find a realization of the sequence.
    a. 5, 5, 5, 3, 3, 2, 2, 2, 2, 2
    b. 7, 6, 5, 5, 4, 3, 2, 2, 2
    c. 4, 4, 3, 2, 1, 0
20. Show that the sequence $d_1, d_2, \cdots, d_p$ is graphical if, and only if, the sequence $p - d_1 - 1, p - d_2 - 1, \ldots, p - d_p - 1$ is graphical.
21. The *degree set* of a graph G is the set of degrees of the vertices of G.
22. a. Show that every set $S = \{a_1, a_2, \ldots, a_k\}$ $(k \geq 1)$ of positive integers with $a_1 < a_2 < \cdots < a_k$ is the degree set of some graph.
    b. Prove that if $u(S)$ is the minimum order of a graph with degree set S, then $u(S) = a_k + 1$.
    c. Find a graph of order 7 with degree set $S = \{3, 4, 5, 6\}$.
23. Show that every graph of order n is isomorphic to a subgraph of K_n.

~~24.~~ Show that every subgraph of a bipartite graph is bipartite.

25. Let G be a bipartite graph. If A_{21} is the transpose of A_{12}, show that the vertices of G can be partitioned so that the adjacency matrix of G has the following form:

$$\begin{bmatrix} 0 & A_{12} \\ A_{21} & 0 \end{bmatrix}$$

26. Let G be a (p, q) graph and let t be an integer, $1 < t < p - 1$. Prove that if $p \geq 4$ and all induced subgraphs of G on t vertices have the same size, then G is isomorphic to K_p or $\overline{K_p}$.

27. Let G be a (p, q) graph. Show that $\delta(G) \leq \frac{2q}{p} \leq \Delta(G)$.

28. Show that the entries on the diagonal of A^2 are the degrees of G.

29. Show that any degree sequence of a graph has two equal terms.

30. Show that $d_1, d_2, \ldots, d_p$ is the degree sequence of a multigraph if and only if $\sum_{i=1}^{p} d_i$ is even.

31. The *line graph* $L(G)$ of a (p, q) graph G is that graph with $V(L(G)) = E(G)$ and such that two vertices in $L(G)$ are adjacent if, and only if, the corresponding edges in G are adjacent. Determine formulas for the order and size of $L(G)$.

32. Find $L(K_{2,3})$.

33. Assuming no human head has more than 2,000,000 hairs on it, show that there are at least two people in New York City with exactly the same number of hairs on their heads.

34. Show that if the digits $1, 2, \ldots, 10$ are used to randomly label the vertices of a C_{10} (no label is repeated), that the sum of the labels on some set of three consecutive vertices along the cycle will be at least 17.

35. Show that there exist graphs on five vertices that do not contain an induced K_3 or $\overline{K_3}$.

36. If we color the vertices of C_{11} either red, white, blue or green, what can be said about the order of the largest subgraph each of whose vertices has the same color?

37. Prove that in any group of $p \geq 2$ people, there are always two people that have the same number of acquaintances.

38. (*) Prove that any sequence of $n^2 + 1$ distinct integers contains either an increasing subsequence of $n + 1$ terms or a decreasing subsequence of $n + 1$ terms.

39. Find a sequence of n^2 $(n \geq 3)$ distinct integers that does not contain an increasing subsequence of $n + 1$ terms or a decreasing subsequence of $n + 1$ terms.

40. Prove that if $n+1$ numbers are selected from the set $\{1, 2, \ldots, 2n\}$, then one of these numbers will divide a second one of these numbers.

Chapter 2

Exercises for Ch. 2

1. Show that graph distance is a metric function. Is distance still a metric function on labeled or weighted graphs?
2. Modify the BFS labeling process to make it easier to find the $x - v$ distance path.
3. Modify the BFS algorithm to find the distance from x to one specified vertex y.
4. Develop a recursive version of the BFS algorithm.
5. What modifications are necessary to make Dijkstra's algorithm work for undirected graphs?
6. Modify Dijkstra's algorithm to find the distance from x to all vertices reachable from x.
7. Prove that the relation "is connected to" is an equivalence relation on the vertex set of a graph.
8. Show that if G is a connected graph of order p, then the size of G is at least $p - 1$.

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Exercises for Ch. 2

9. Determine a sharp lower bound on the size q of a (p, q) graph necessary to ensure that the graph is connected.
10. Characterize those graphs having the property that every one of their induced subgraphs is connected.
11. Prove that every circuit in a graph contains a cycle.
12. Prove that if G is a graph of order p and $\delta(G) \geq \frac{p}{2}$, then $k_1(G) = \delta(G)$.
13. Suppose that G is a (p, q) graph with $k(G) = n$ and $k_1(G) = m$, where both n and m are at least 1. Determine what values are possible for the following:
$$k(G - v),\ k_1(G - v),\ k(G - e),\ k_1(G - e).$$
14. Let G be an n-connected graph and let $v_1, v_2, \ldots, v_n$ be distinct vertices of G. Suppose we insert a new vertex x and join x to each of $v_1, v_2, \ldots, v_n$. Show that this new graph is also n-connected.
15. Prove that if G is an n-connected graph and $v_1, \ldots, v_n$ and v are $n + 1$ vertices of G, then there exist internally disjoint $v - v_i$ paths for $i = 1, \ldots, n$.
16. Show that if G contains no vertices of odd degree, then G contains no bridges.
17. Prove Theorem 2.1.4.
18. In applying Ford's algorithm to a weighted digraph D that contains no negative cycles, show that if a shortest $x - v$ path contains k arcs, then v will have its final label by the end of the kth pass through the arc list.
19. Modify the labeling in Ford's algorithm to make backtracking to find the distance path easier.
20. Show that G contains a path of length at least $\delta(G)$.
21. Show that G is connected if, and only if, for every partition of $V(G)$ into two nonempty sets V_1 and V_2, there is an edge from a vertex in V_1 to a vertex in V_2.
22. Show that if $\delta(G) \geq \frac{p-1}{2}$, then G is connected.
23. Show that any nontrivial graph contains at least two vertices that are not cut vertices.

Exercises for Ch.2

24. Show that if G is disconnected, then $\overline{G}$ is connected.

25. Show that if G is connected, then either G is complete or G contains three vertices x, y, z such that xy and yz are edges of G but $xz \notin E(G)$.

26. A graph G is a *critical block* if G is a block and for every vertex v, $G - v$ is not a block. Show that every critical block of order at least 4 contains a vertex of degree 2.

27. A graph G is a *minimal block* if G is a block and for every edge e, $G - e$ is not a block. Show that if G is a minimal block of order at least 4, then G contains a vertex of degree 2.

28. The *block index* $b(v)$ of a vertex v in a graph G is the number of blocks of G to which v belongs. If $b(G)$ denotes the number of blocks of G, show that

$$b(G) = \kappa(G) + \sum_{v \in V(G)} (b(v) - 1).$$

29. Three cannibals and three missionaries are traveling together and they arrive at a river. They all wish to cross the river; however, the only transportation is a boat that can hold at most two people. There is another complication, however; at no time can the cannibals outnumber the missionaries (on either side of the river), for then the missionaries would be in danger. How do they manage to cross the river?

30. Prove that there can be no solution to the three cannibals and three missionaries problem that uses fewer than eleven river crossings.

31. Does a four-cannibal and four-missionary problem make sense? If so, explain this problem and try to solve it.

32. Three wives and their jealous husbands wish to go to town, but their only means of transportation is an RX7, which seats only two people. How might they do this so that no wife is ever left with one or both of the other husbands unless her own husband is present?

33. The 8-puzzle is a square tray in which are placed eight numbered tiles. The remaining ninth square is open. A tile that is adjacent to the open square can slide into that space. The object of this game is to obtain the following configuration from the starting configuration:

Exercises for Ch. 2

#33. (continued)

start

2	8	3
1	6	4
7		5

goal

1	2	3
8		4
7	6	5

How does this problem differ from those studied earlier? Can you build a mechanism into the rules that handles this difference?

34. A problem-solving search can proceed forward (as we have done) or backward from the goal state. What factors should influence your decision on how to proceed?

Chapter 3

Exercises Exercises for Chapter 3

1. Show that if $T = (V, E)$ is a tree, then for any $e \in E$, $T - e$ has exactly two components.

2. Show that any connected graph on p vertices contains at least $p - 1$ edges.

3. Show that if T is a tree with $\delta(T) \geq k$, then T has at least k leaves.

4. In a connected graph G, a vertex v is called *central* if $\max_{u \in V(G)} d(u, v) = rad(G)$. Show that for a tree T, the set of central vertices consists of either one vertex or two adjacent vertices.

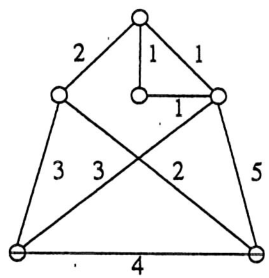
5. Show that the sequence $d_1, d_2, \ldots, d_p$ of positive integers is the degree sequence of a tree if, and only if, the graph is connected and $\sum_{i=1}^{p} d_i = 2(p-1)$.

Exercises for Ch. 3

6. Show that the number of end vertices in a nontrivial tree of order n equals $2 + \sum_{deg\ v_i \geq 3} (\ deg\ v_i - 2\ )$.

7. Determine the time complexity of Kruskal's algorithm.

8. Apply Kruskal's algorithm to the graph:



9. Apply Prim's algorithm to the graph of the previous problem.

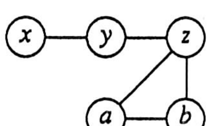
10. Prove that a graph G is acyclic if, and only if, every induced subgraph of G contains a vertex of degree one at most.

11. Characterize those graphs with the property that every connected subgraph is also an induced subgraph.

12. Find the binary tree representations for the expressions $4x - 2y$, $(\ 3x + z\ )(\ xy - 7z\ )$, and $\sqrt{b^2 - 4ac}$.

13. Perform a preorder, postorder and inorder traversal on the trees constructed in the previous problem.

14. Determine the number of nonidentical spanning trees of the graph below. Before you begin your computation, make an observation about this graph that will simplify the calculations.



15. Find the spanning trees on the set { 1, 2, 3, 4 }.

16. Using the proof of Cayley's theorem, determine the sequences of length $p - 2$ on { 1, 2, 3, 4 } that correspond to any two of the trees found in the previous problem.

Exercises for Ch. 3

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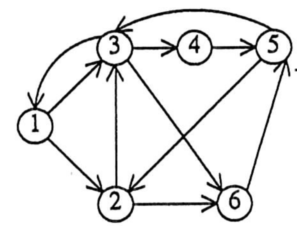
17. Show that the Prüfer algorithm for creating a tree from a sequence selects the same vertex as the algorithm for producing the sequence.

18. Use the matrix-tree theorem to prove Cayley's tree formula.

19. Find the number of directed spanning trees with root 3 in the following digraph:



20. Show that the number of trees with m labeled edges and no labels on the vertices is $(\ m + 1\ )^{m-2}$.

21. Determine the number of trees that can be built on p labeled vertices such that one specified vertex is of degree k.

22. By *contracting an edge* $e = uv$, we mean removing e and identifying the vertices u and v as a single new vertex. Let $num_T(\ G\ )$ denote the number of spanning trees of the graph G. Show that the following recursive formula holds:

$$num_T(\ G\ ) = num_T(\ G - e\ ) + num_T(\ G \circ e\ )$$

where $G \circ e$ means the graph obtained from G by contracting the edge e. Hint: Interpret what $num_T(\ G - e\ )$ and $num_T(\ G \circ e\ )$ really count.

23. Show that the algorithm for determining the number of spanning trees of G implied by the previous problem takes exponential time.

24. Determine the Huffman tree and code for the alphabet $\{\ x,\ y,\ z,\ a,\ b,\ c\ \}$ with corresponding frequencies (3, 8, 1, 5, 4, 4).

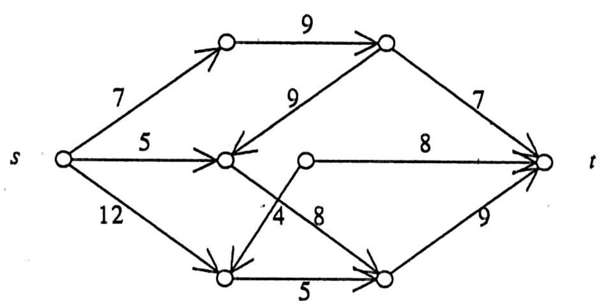
25. What is the minimum weighted path length for the Huffman tree you constructed in the previous problem?

26. Using the definition of $\binom{1/2}{n+1}$, show that

$$\binom{1/2}{n+1} = \frac{(-1)^n}{2^{n+1}} \times \frac{(\ 1\ )(\ 3\ )(\ 5\ )\cdots(\ 2n - 1\ )}{(\ 1\ )(\ 2\ )(\ 3\ )\cdots(\ n + 1\ )} = \frac{(2n)!\,(-1)^n}{2^{2n+1}\,n!\,(n+1)!}$$

Exercises for Chapter 4

1. Find the maximum flow in the network below, using each of the algorithms presented in this chapter.



2. Use Algorithm 4.3.2 to find the maximum flow in the network of Figure 4.1.2.

3. Use Algorithm 4.4.1 to find the maximum flow in the network of Figure 4.1.2.

4. Let u and v be two vertices of a digraph $D = (V, E)$ and let $E_1 \subseteq E$ such that every $u - v$ path in D contains at least one arc of E_1. Prove that there exists a set of arcs of the form $(W, \overline{W})$ (where $W \subseteq V$) such that $u \in W, v \in \overline{W}$ and $(W, \overline{W}) \subseteq E_1$.

5. Suppose the following network has multiple sources s_1, s_2 and s_3 and they have available supplies 6, 12, and 7, respectively. Further suppose that t_1, t_2 and t_3 are all sinks with the demands 7, 12 and 6, respectively. Determine whether all these demands can be met simultaneously.

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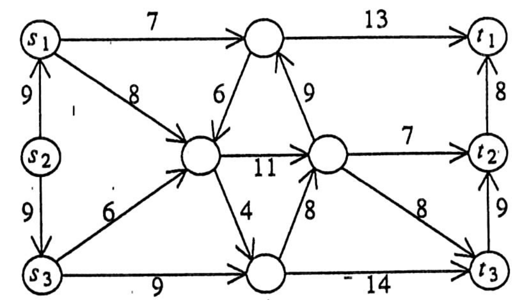
Exercises for Ch. 4

#5.

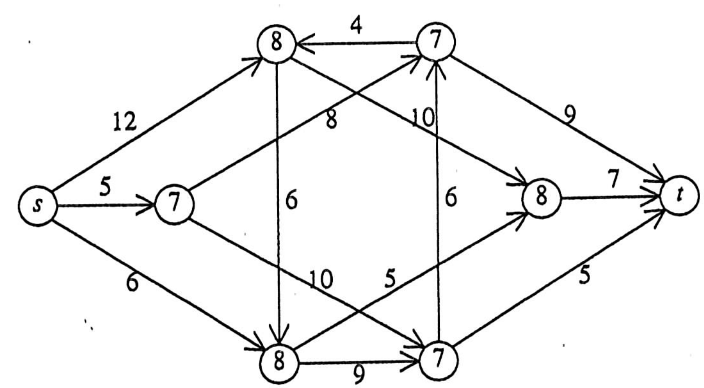


6. Let N be a network with underlying digraph D. Prove that if D contains no $s - t$ path, then the value of a maximum flow in N and the value of a minimum cut in N are both equal to zero.

7. Prove or disprove (each direction): The flows f_1 and f_2 in the network N agree on $(W, \overline{W})$ and $(\overline{W}, W)$ if, and only if, f_1 and f_2 are maximum flows in N.

8. Use networks to prove that $k(K_{m,n}) = \min\{m, n\}$.

9. In the following network, in addition to the usual constraints, we have the additional constraint that each vertex (other than s and t) has an upper bound on the capacity that may flow through it. This capacity is indicated by the value inside the vertex. Find the maximum flow for this network.



10. Prove that in a network with a nonnegative lower bound on each arc, but no upper bound, there exists a legal flow if, and only if, for every arc e, either e is in a directed circuit or e is in a directed path from s to t (or t to s).

11. Prove that a network with both lower and upper bounds on the flow in the arcs has no legal flow if, and only if, there exists a set of vertices which includes neither the source nor the sink and is required to produce flow or absorb it.

12. A *path cover* of the vertices of a digraph $D = (V, E)$ (or graph) is a set of paths with the property that each vertex of D lies on exactly one path (note that a path may be trivial). Use the following network to develop an algorithm for determining the minimum number of paths necessary to cover an acyclic digraph $D = (V, E)$: Let $N = (V_1, E_1)$ where

$$V_1 = \{s, t\} \cup \{u_1, \ldots, u_{|V|}\} \cup \{v_1, \ldots, v_{|V|}\}$$

$$E_1 = \{s \to u_i \mid 1 \le i \le |V|\} \cup \{v_i \to t \mid 1 \le i \le |V|\} \cup \{u_i \to v_j \mid x_i \to x_j \in E(D)\}.$$

Also, set the capacities of each arc to 1. (Hint: Show that the minimum number of paths which cover V equals $|V|$, the max flow in the network.)

13. In the previous problem, was the assumption that D was acyclic necessary?

14. Suppose we ease the path cover restriction in the sense that the paths are no longer required to be vertex disjoint. Further, into the network constructed in exercise 11, insert the additional edges $\{u_i \to v_i \mid 1 \le i \le |V|\}$. Now, let the lower bound on each of these new arcs be 1 and on all other arcs be 0 and let all upper bounds be ∞. Describe an algorithm for finding the minimum number of paths covering the vertices of the original digraph D.

15. (Dilworth's theorem [3]) Two vertices are called *concurrent* if no directed path exists between them. A set of vertices is *concurrent* if its members are pairwise concurrent. Prove that the minimum number of paths which cover the vertices of D equals the maximum number of concurrent vertices. (Hint: See the previous exercise.)

16. Verify that the modified Ford and Fulkerson algorithm for networks with upper and lower bounds actually gives the desired results.

17. Complete the proof of the edge version of Menger's theorem (Theorem 4.6.2).

18. If $paths(u, v)$ is the maximum number of pairwise disjoint $u - v$ paths in a graph G, show that if G is not complete, then

$$\min_{u,v \in V(G)} paths(u, v) = \min_{uv \notin E(G)} paths(u, v);$$

that is, the minimum value of $paths(u, v)$ occurs for some nonadjacent pair of vertices u, v.

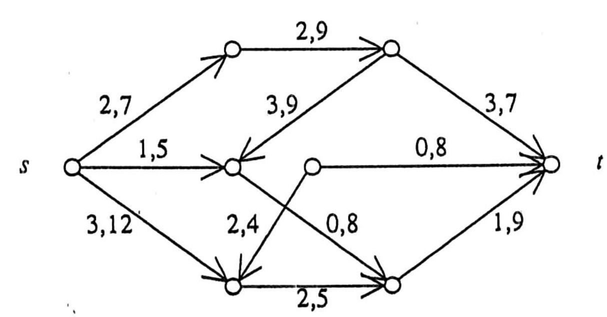
19. Show that if the connectivity of a graph G is k, then

$$k = \min_{u,v \in V(G)} paths(u, v).$$

20. Let N be a $0-1$ network with no multiple arcs and let M be the maximum total flow from s to t in N. Show that the length of the first layered network (with zero flow everywhere) is at most $\frac{2|V|}{M^{1/2}} + 1$.

21. Show that for the network N of the previous problem, Dinic's algorithm has time complexity $O(|V|^{2/3} |E|)$.

22. Determine if the network below has a legal flow. If it does not, modify as few capacities as possible to obtain a network that does have a legal flow.



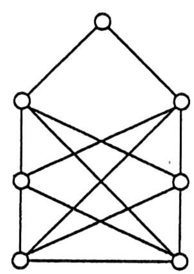
Exercises for Chapter 5

1. Prove Corollary 5.1.2.

2. Prove Theorem 5.1.2.

3. Determine the complexity of Algorithm 5.1.1.

4. Use each of Algorithms 5.1.1, 5.1.2 and 5.1.3 to find an eulerian cycle in the graph below.



5. Determine an algorithm to accomplish the splitting away of two edges.

6. If H is the graph obtained from G by splitting away $e_1 = vw$ and $e_2 = vx$, prove that H is connected if, and only if, G is connected and $\{ e_1, e_2 \}$ does not form a cut set.

7. Prove that a nontrivial connected graph G is eulerian if, and only if, every edge of G lies on an odd number of cycles.

8. Prove that a nontrivial connected digraph D is eulerian if and only if E can be partitioned into subsets $E_1, E_2, \ldots, E_k$ such that the graph induced by E_i is a cycle for each i, $(1 \le i \le k)$.

9. Prove Observations 1 and 2 from page 135.

10. Show that if $G = (V, E)$ is hamiltonian, then for every proper subset S of V, the number of components in $G - S$ is at most $|S|$.

11. Show that if G is not 2-connected, then G is not hamiltonian.

12. Characterize when the graph $K_{p_1, p_2, \ldots, p_n}$ is hamiltonian.

13. Prove or disprove: If G and H are hamiltonian, then $G \times H$ is hamiltonian.

14. Prove or disprove: If G and H are hamiltonian, then $G[H]$ is hamiltonian.

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Exercises for Ch. 5

15. Let the n-cube be the graph $Q_n = K_2 \times Q_{n-1}$ (where $Q_1 = K_2$). Prove that if $n \ge 2$, Q_n is hamiltonian.

16. Let G be a graph with $\delta(G) \ge 2$. Show that G contains a cycle of length at least $\delta(G) + 1$.

17. Suppose that G is a (p, q) graph with $p \ge 3$. Show that if $q \ge \dfrac{p^2 - 3p + 6}{2}$, then G is hamiltonian.

18. Show that if G is a (p, q) graph with $q \ge \binom{p-1}{2} + 3$, then G is hamiltonian connected.

19. Show that if G is hamiltonian connected, then G is 3-connected.

20. Show that if a (p, q) graph G is hamiltonian connected and if $p \ge 4$, then $q \ge \left\lceil \dfrac{3p+1}{2} \right\rceil$.

21. Give an example of a graph that is pancyclic but not panconnected. (Hint: Consider order 8.)

22. Find an example of a graph that is hamiltonian connected but not panconnected.

23. Find an example of a graph that is pancyclic but not vertex pancyclic.

24. Can we remove the restriction that D be strongly connected from Meyniel's theorem?

25. Show that every complete graph with directed edges is traceable.

26. Show that K_n with strong directed edges is vertex pancyclic.

27. Give an example of a hamiltonian connected digraph that satisfies the conditions of Theorem 5.4.2 but does not have $od\ v \ge \dfrac{p+1}{2}$ and $id\ v \ge \dfrac{p+1}{2}$ for every vertex v.

28. Show that the Petersen graph is homogeneously traceable nonhamiltonian and also hypohamiltonian.

29. Show that homogeneously traceable nonhamiltonian graphs exist for all orders $p \ge 9$.

30. Show that if $G = (V, E)$ is a homogeneously traceable nonhamiltonian graph and $x \in V$, then x is adjacent to at most one vertex of degree 2.

Exercises for Ch. 5

31. Show that if $C_{p-1}(G) = K_p$, then G is traceable.

32. Prove Corollary 5.4.2.

33. Prove that the graph G^2 of Figure 5.5.1 is not hamiltonian.

34. Show (without using Fleischner's theorem) that if G is 2-connected, then G^3 is hamiltonian. (Hint: Consider spanning trees).

35. Use Fleischner's theorem to show that if G is 2-connected, then G^2 is hamiltonian connected. (Hint: Consider five copies of G along with two additional vertices x and y joined to an arbitrary pair of vertices u and v in each copy of G).

36. Prove Theorem 5.5.2.

37. Prove that if G is a graph of order $p \geq 3$ such that the vertices of G can be labeled $v_1, v_2, \ldots, v_p$ so that

$$\begin{matrix} j<k,\ j+k \geq p,\ v_j v_k \notin E(G) \\ \deg v_j \leq j,\ \deg v_k \leq k-1 \end{matrix} \Rightarrow \deg v_j + \deg v_k \geq p.$$

then G is hamiltonian.

38. Let G be a graph of order $p \geq 3$, the degrees d_i of whose vertices satisfy $d_1 \leq d_2 \leq, \ldots, \leq d_p$. If

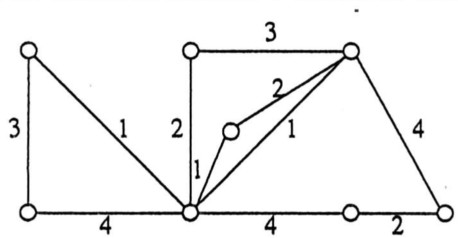
$$d_j \leq j < \frac{p}{2} \Rightarrow d_{p-j} \geq p-j,$$

then G is hamiltonian.

39. Prove that if G is a graph of order $p \geq 3$ such that for every integer j with $1 \leq j < \frac{p}{2}$, the number of vertices of degree not exceeding j is less than j, then G is hamiltonian.

40. Prove that if G has order $p \geq 3$ and if $k(G) \geq \beta(G) =$ the maximum number of mutually nonadjacent vertices, then G is hamiltonian.

41. Determine the minimum salesman's walk in the following graph.



Exercises for Ch. 5

42. Show that the greedy approach to the traveling salesman problem can be arbitrarily bad.

43. Prove that the graph of Figure 5.7.1 is the unique (5, 3)-cage and is isomorphic to the Petersen graph.

44. Find the (6, 3)-cage and show that it is unique.

45. Prove Theorem 5.7.4(1).

Chapter 6

Exercises for Chapter 6.

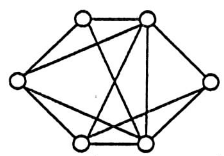
1. Show that if G is a plane (p, q) graph with r regions, then $p - q + r = 1 + k(G)$ where $k(G)$ is the number of components of G.

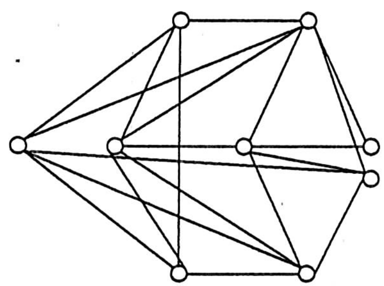
Exercises for Ch. 6

2. An (f, d)– *regular polyhedron graph* is a plane graph that is d-regular $(d \geq 3)$ and each of its faces has f sides. Use Euler's formula to show that there are only five regular polyhedron graphs. (Hint: The dodecahedron is a $(5, 3)$–regular polyhedron graph.)
3. Show that "homeomorphic with" is an equivalence relation on the set of graphs.
4. Show that a graph is planar if, and only if, each of its blocks (maximal 2-connected subgraphs) is planar.
5. Let G be a maximal planar graph of order $p \geq 4$. Also let p_i denote the number of vertices of degree i in G, where $i = 3, 4, \ldots, \Delta(G) = n$. Show that
$$3p_3 + 2p_4 + p_5 = p_7 + 2p_8 + \cdots + (n-6)p_n + 12.$$
6. Show that any maximal planar (p, q) graph contains a bipartite subgraph with $2\frac{q}{3}$ edges.
7. Find an example of a planar graph that contains no vertex of degree less than 5.
8. Prove that every planar graph of order $p \geq 4$ contains at least four vertices of degree at most 5.
9. Show that the Petersen graph (Figure 7.3.2) contains a subgraph homeomorphic with $K_{3,3}$ and is therefore not planar.
10. Show that if G is a connected planar (p, q) graph with girth (shortest cycle length) $g(G) = k \geq 3$, then $|E| \leq \frac{k(p-2)}{(k-2)}$.
11. Use the last result to again show that the Petersen graph is not planar.
12. A graph is self-dual if it is isomorphic to its own geometric dual. Show that if G is self-dual, then $2|V| = |E| + 2$. Further, show that not every graph with this property is self-dual.
13. If G is a connected plane graph with spanning tree T and $E^* = \{e^* \in E(G^*) \mid e \notin E(T)\}$, show that $T^* = \left(E^*\right)$ is a spanning tree of G^*.
14. Show that if $|V(G)| \geq 11$, then at least one of G and $\overline{G}$ is nonplanar.
15. Show that the average degree in a planar graph is actually less than 6. (Note that this provides an alternate proof to Corollary 6.1.2.)

Exercises for Ch. 6

16. Use the DMP algorithm to test the planarity of the following graphs.





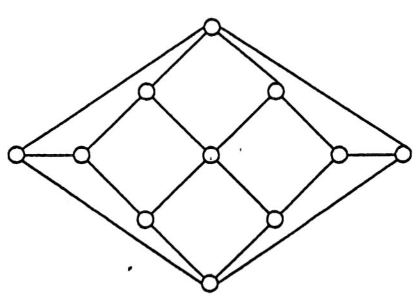
17. Prove path properties 1–5 relating to the Hopcroft-Tarjan planarity algorithm.

18. Use the Hopcroft-Tarjan planarity algorithm to test the planarity of the graphs from exercise 16 in Chapter 6.

19. Prove Theorem 6.4.1.

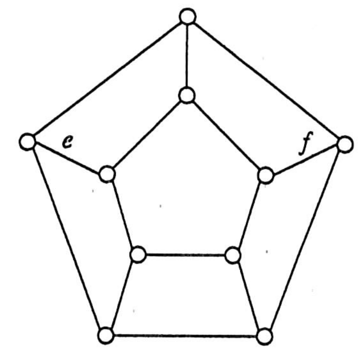
20. How might you actually keep track of the paths both inside and outside of a given cycle? How much information must actually be recorded?

21. Use Grinberg's theorem to show that the graph below is not hamiltonian.



Exercises for Ch. 6

22. Show that no hamiltonian cycle in the graph below can contain both of the edges *e* and *f*.



Chapter 7

B 12

Exercises for Ch. 7

27. Find a decomposition of K_{10} as paths of length 5.

28. Prove that for each integer $n \geq 1$, the graph K_{2n+1} can be decomposed as a collection of stars $K_{1,n}$ and that the graph K_{2n} can be decomposed as a collection of stars $K_{1,n}$.

Exercises for Chapter 7

1. Show that the n-cube Q_n ($n \geq 2$) has a perfect matching.

2. Show that Q_n is r-factorable if, and only if, $r \mid n$.

3. Characterize when the graph $K_{p_1, p_2, \ldots, p_n}$ has a perfect matching.

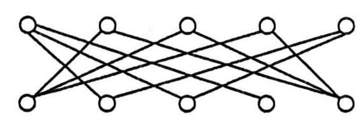
4. Determine the number of perfect matchings in the graphs $K_{p,p}$ and K_{2p}.

5. How many perfect matchings can exist in a tree?

6. Find a maximum matching and a minimum cover in the graph below using each of the indicated methods.

   a. Algorithm 7.2.1 and Theorem 7.2.1.

   b. A network model.



7. Use Dirac's theorem (Corollary 5.2.1) to show that if G has even order and $\delta(G) \geq \frac{p}{2} + 1$, then G has a 3-factor.

8. Show that every doubly stochastic matrix is a square matrix.

9. Show that if $G = (X \cup Y, E)$ is a bipartite graph, then
$$\beta_1(G) = |X| - \max_{S \subseteq X} \{|S| - |N(S)|\}.$$

10. Use the previous exercise to show that if the (p, q) graph $G = (X \cup Y, E)$ is bipartite and $|X| = |Y| = n$ and $q \geq (k-1)n$, then G has a matching of cardinality k.

11. [9] Suppose that G is a graph of order p with the property that for every pair of nonadjacent vertices x and y, $|N(x) \cup N(y)| \geq s$.
   a. Use Berge's defect form of Tutte's theorem to show that if

11. $s > 2\left\lfloor \frac{p}{3} \right\rfloor - 2$ and p is odd and $p \geq 6$, then

$$\beta_1(G) = \frac{1}{2}(p-1).$$

b. Find a graph of order 5 for which the conditions of part (a) fail to ensure $\beta_1(G) = \frac{1}{2}(p-1)$.

c. Use Tutte's theorem to show that if $s > \frac{2}{3}(p-1) - 1$ and p is even and G is connected, then $\beta_1(G) = \frac{p}{2}$.

12. Use Tutte's theorem to prove Hall's theorem.

13. Use König's theorem to prove Hall's theorem.

14. Prove Corollary 7.3.1.

15. Prove Theorem 7.3.6.

16. Show that K_{2n} can be factored into $n-1$ hamiltonian paths and one 1-factor.

17. Let G be a (p, q) graph of even order p with $\delta(G) < \frac{p}{2}$. Show that if

$$q > \binom{\delta(G)}{2} + \binom{p - 2\delta(G) - 1}{2} + \delta(G)(p - \delta(G)),$$

then G has a perfect matching.

18. Four men and four women apply to a computer dating service. The computer evaluates the unsuitability of each man for each woman as a percentage (see the table below). Find the best possible dates for each woman for this Friday night.

	M_1	M_2	M_3	M_4
W_1	60	35	30	65
W_2	30	10	55	30
W_3	40	60	15	35
W_4	25	15	40	40

19. Consider the table used for the last exercise as representing the weights assigned to a bipartite graph and solve the bottleneck assignment problem for this graph.

20. The math department at your college has six professors that must be assigned to teach each of five different classes. The department did an examination of the suitability of each professor for each class and the unsuitability table is shown below. What is the optimal teaching assignment that can be made if no professor is assigned more than one class?

	P_1	P_2	P_3	P_4	P_5	P_6
C_1	75	25	55	25	50	35
C_2	60	30	45	35	45	20
C_3	55	25	50	15	50	30
C_4	40	35	40	45	35	25
C_5	50	20	45	30	40	45

(Hint: Add a dummy class that each professor is equally suited to teach.)

21. Does the previous problem make sense as a bottleneck assignment problem? If so, solve it.

22. Consider the doubly stochastic matrix below. Use Algorithm 7.2.2 to decompose this matrix into permutation matrices.

$$\begin{vmatrix} 0.3 & 0.3 & 0.0 & 0.3 & 0.1 \\ 0.1 & 0.5 & 0.2 & 0.1 & 0.1 \\ 0.2 & 0.0 & 0.3 & 0.5 & 0.0 \\ 0.0 & 0.2 & 0.5 & 0.0 & 0.3 \\ 0.4 & 0.0 & 0.0 & 0.1 & 0.5 \end{vmatrix}$$

23. Consider the table of the previous problem as the weights assigned to the edges of a bipartite graph. Interpret your solution in relation to the last problem on this graph.

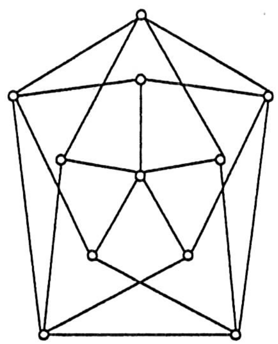
24. Explain why the adjustment process allows us to complete the hungarian algorithm applied to an unsuitability matrix.

25. A *decomposition* of G is a collection $\{H_i\}$ of subgraphs of G such that $H_i = LG\ E_i\ RG$ for some subset E_i of $E(G)$ and where the sets $\{E_i\}$ partition $E(G)$. Prove that the complete graph K_p can be decomposed as a collection of 3-cycles if, and only if, $p \geq 3$, p is odd and 3 divides $\binom{n}{2}$.

26. Find a decomposition of K_5 as 5-cycles.

Exercises for Ch. 8

26. a. $K_{1,3}$ ← #26 (continued)
    b. $K_4 - e$
    c. The Petersen graph
    d. The Grötsch graph shown below



27. Find the chromatic polynomial for $K_{1,3}$ and for $K_4 - e$. How many 5-colorings are there for each of these graphs?

28. Let G be a graph. For every nontrivial subset S of $V(G)$, either $<S>_G$ or $<S>_{\bar{G}}$ is disconnected if, and only if, G is P_4-free.

Exercises for Chapter 8

1. Show that $\chi_1(K_{m,n}) = \max\{m, n\}$.
2. Show that if G is a bipartite graph, then $\chi_1(G) = \delta(G)$.
3. Prove that if G is a graph of order p with $\delta(G) > 0$, then $\alpha_1(G) + \beta_1(G) = p$.
4. Prove that $\chi(K_{p_1, \ldots, p_n}) = n$.
5. Prove that if G is k-partite, then $\chi(G) \leq k$.
6. Prove Corollary 8.2.1.
7. Prove Corollary 8.2.2.
8. Prove Corollary 8.2.3.

Exercises for Ch. 8

9. Prove Theorem 8.3.1.
10. Show that every k-chromatic graph is a subgraph of some complete k-partite graph.
11. Determine the n-critical graphs for $n = 1, 2, 3$.
12. Show that a critically n-chromatic graph need not be $(n-1)$-connected.
13. Characterize graphs whose line graphs are 2-colorable.
14. Show that for every graph G, $\chi(G) \leq 1 + \max \delta(H)$ where the maximum is taken over all induced subgraphs H of G.
15. If $m(G)$ denotes the length of a longest path in G, prove that $\chi(G) \leq 1 + m(G)$.
16. Find a largest first ordering of the vertices of the graph in Example 8.3.1 that produces a sequential coloring using three colors.
17. Show that every regular graph of odd order is class 2.
18. Show that if H is a regular graph of odd order and if G is any graph obtained from H by deleting at most $\frac{1}{2}\delta(G) - 1$ edges, then G is of class 2.
19. Show that if H is a regular graph of even order and if G is any graph obtained from H by subdividing any edge of H, then G is class 2.
20. Show that if G is any graph obtained from an odd cycle C_{2k+1} by adding no more than $2k - 2$ independent edges, then G is class 2.
21. Show that if G is a regular graph containing a cut vertex, then G is of class 2.
22. Show that there are no regular $\delta(G)$-minimal graphs with $\delta(G) \geq 3$.
23. Show that every bipartite graph is perfect.
24. Let $G_1, G_2, \ldots, G_k$ be pairwise disjoint graphs. Also let $G = G_1 + G_2 + \cdots + G_k$. Prove that $\chi(G) = \sum \chi(G_i)$ and that $\omega(G) = \sum_{i=1}^{k} \omega(G_i)$.
25. Show that if G does not contain P_4 as an induced subgraph then $\chi(G) = \omega(G)$. Are such graphs perfect?
26. Use the largest first, smallest last and color-degree algorithms to bound the chromatic number of each of the following graphs.