MAD 3305 - GRAPH THEORY

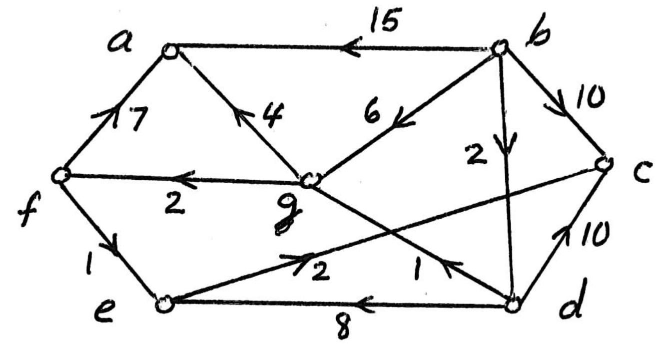
FLORIDA INT'L UNIV.

TEST #1 - SPRING 2006

TIME: 75 min.

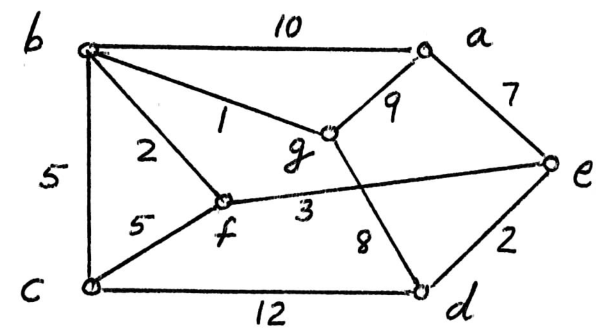
Answer *all* 6 questions. Provide all reasoning and show all working. *An unjustified answer will receive little or no credit.* *BEGIN EACH OF THE 6 QUESTIONS ON A SEPARATE PAGE.*

(15) 1. Find the distances from b to each of the other vertices of the graph on the right by using *Dijkstra's Algorithm*.



(20) 2(a) Determine whether or not the seq. 6,5,4,4,4,3,2 is *graphical*.

(b) Find a minimal spanning tree of the graph on the right by using *Prim's Algorithm* & starting at c.



(20) 3. (a) Find the tree corresponding to $\langle 2,4,1,2,5\rangle$ via *Prufer's Tree Decoding Algorithm*.

(b) The seven characters a,b,c,d,e,f,g occur with frequencies 22,4,5,9,35,10,15 respectively. Find an optimal binary coding for the seven characters.

(15) 4. (a) Define what it means for two graphs G and H to be *isomorphic*.

(b) Let G be a non-trivial graph. Prove that we can always find two vertices in G which have the same degree.

(15) 5. (a) Define what is the *height* of a rooted tree.

(b) Let T be any binary tree with p vertices. Prove that

$$h(T) \geqslant \log_2[(p+1)/2].$$

(15) 6. (a) Define what is a *pendant vertex* and what is a *leaf*.

(b) A certain tree T has 10 vertices of degree 5, 20 of degree 4, 30 of degree 3 and the rest of degree 1 or 2. How many leaves does T have ?

[*You may use any theorem proved in class in Qu.#6, if needed.*]

MAD 3305 – Graph Theory — **Florida Int'l Univ.**

Solutions to Test #1 — **Spring 2006.**

1.

	L(a)	L(b)	L(c)	L(d)	L(e)	L(f)	L(g)	T	o
	∞	$\underline{0}$	∞	∞	∞	∞	∞	$\{a,b,c,d,e,f,g\}$	b
	15	.	10	$\underline{2}$	∞	∞	6	$\{a,c,d,e,f,g\}$	d
	15	.	10	.	10	∞	$\underline{3}$	$\{a,c,e,f,g\}$	g
	7	.	10	.	10	$\underline{5}$	.	$\{a,c,e,f\}$	f
	7	.	10	.	$\underline{6}$	.	.	$\{a,c,e\}$	e
	$\underline{7}$	.	8	.	.	.	.	$\{a,c\}$	a
	.	.	$\underline{8}$	.	.	.	.	$\{c\}$	c
	.	.	.	.	.	.	.	$\emptyset$	
$d(b,\cdot)$	7	0	8	2	6	5	3		

2(a) 6, 5, 4, 4, 4, 3, 2

4, 3, 3, 3, 2, 1

2, 2, 2, 1, 1

1, 1, 1, 1 → 1, 1, 1, 1

0, 1, 1

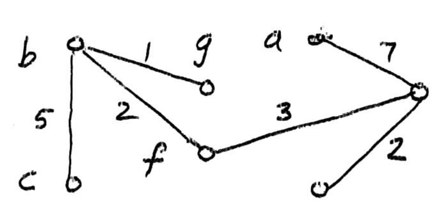
1, 1, 0

0, 0. $\therefore$ Graphical

(b)

E(T)	U	a	b	c	d	e	f	g	x_0
$\emptyset$	$\{c\}$	∞	$\underline{5}$	.	12	∞	5	∞	b
$+\{cb\}$	$\{c,b\}$	10	.	.	12	∞	2	$\underline{1}$	g
$+\{bg\}$	$\{c,b,g\}$	9	.	.	8	∞	$\underline{2}$	.	f
$+\{bf\}$	$\{c,b,g,f\}$	9	.	.	8	$\underline{3}$	.	.	e
$+\{fe\}$	$\{c,b,g,f,e\}$	7	.	.	$\underline{2}$	.	.	.	d
$+\{ed\}$	$\{c,b,g,f,e,d\}$	$\underline{7}$	.	.	.	.	.	.	a
$+\{ea\}$	$\{c,b,g,f,e,d,a\}$								

T =

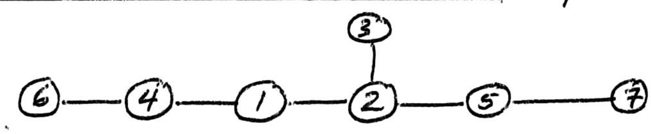


3 (a)

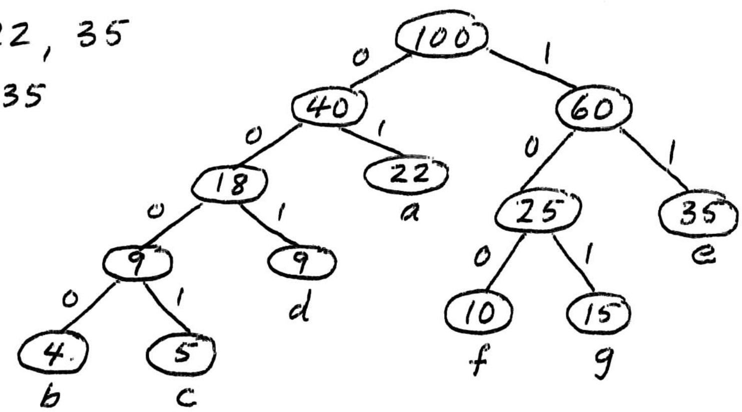
degrees each vertex needs at stage i

i \ x	①	②	③	④	⑤	⑥	⑦	$j(i)$	$s(i)$
1	2	3	1	2	2	1	1	3	2
2	2	2	0	2	2	1	1	6	4
3	2	2	·	1	2	0	1	4	1
4	1	2	·	0	2	·	1	1	2
5	0	1	·	·	2	·	1	2	5
	·	0	·	·	1	·	1	plus 5	7

$\therefore$ $T =$



(b) $\underline{4}, \underline{5}, 9, 10, 15, 22, 35$
$\underline{9}, \underline{9}, 10, 15, 22, 35$
$10, 15, 18, 22, 35$
$18, 22, 25, 35$
$25, 35, 40$
$40, 60$
$100.$



Char.	a	b	c	d	e	f	g
Code	01	0000	0001	001	11	100	101

4(a) G and H are said to be isomorphic if there exists a bijection $\alpha: V(G) \to V(H)$ such that $uv \in E(G) \Leftrightarrow \alpha(u)\alpha(v) \in E(H)$.

(b) Let G be a non-trivial graph and $p = |V(G)|$. Then $p \geq 2$. Now either G has a vertex of degree 0 or all vertices in G are of degree ≥ 1.

Case (i): G has a vertex of degree 0.
In this case the maximum degree in G will be at most $p-2$. So the possible degrees are $0, 1, 2, \ldots, p-2$.

4(b) Since G has p vertices and there are only $p-1$ possible degrees we must have two vertices with the same degree.

Case (ii): All vertices in G are of degree ≥ 1.
In this case the possible degrees in G are $1, 2, 3, \ldots, p-1$ because the max. deg. in G will be at most $p-1$. Since G has p vertices and there are only $p-1$ possible degrees we again must have two vertices with the same degree.
So in either case we can find two vertices with the same degree.

5(a) The height of a rooted tree T is the distance from the root to the furthermost vertex in T. You can also say it is the highest level that exists in T.

(b) Since T is a binary tree, level i will have at most 2^i vertices. So $p \leq 1 + 2^1 + 2^2 + \cdots + 2^k$ where $k = h(T)$. Hence $p \leq (2^{k+1} - 1)/(2-1)$. So $p \leq 2^{k+1} - 1$. $\therefore$ $2^{k+1} \geq p+1$. So $2^k \geq \frac{p+1}{2}$.
Thus $h(T) = k \geq \log_2 \left(\frac{p+1}{2}\right)$.

6(a) A pendant vertex is a vertex of degree 1 in a graph.
A leaf is a pendant vertex in a tree.

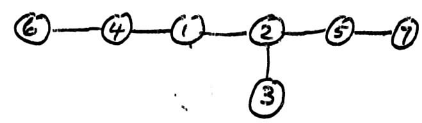
(b) Let X and Y be the no. of vertices of degree 1 and 2 respectively. Then $p = 10 + 20 + 30 + X + Y = 60 + X + Y$ and sum of the degrees in $T = 10(5) + 20(4) + 30(3) + 2Y + X$ $= 220 + 2Y + X$. But sum of degrees $= 2|E| = 2(p-1)$.

$\therefore$ $2(60 + X + Y - 1) = 220 + 2Y + X$. $\therefore$ $120 + 2X + 2Y - 2 = 220 + X + 2Y$

$\therefore$ $X = 220 + 2 - 120 = 102$. So T has 102 leaves.

In the 3 previous pages, the simplest or most straight forward solutions were presented. Other solutions exist but they are more complicated (and sometimes shorter).

3(a) Write down the sequence $\underset{\sim}{s}$ and below it, write all the vertices that do not occur in $\underset{\sim}{s}$. (T will be a tree on the vertices $1, 2, 3, \ldots, |\underset{\sim}{s}|+2$.) Join $s(1)$ with the smallest vertex in the 2nd list. Then delete $s(1)$ from $\underset{\sim}{s}$ to get $\underset{\sim}{s}'$ and delete the smallest vertex just used. Also as you go along, don't forget to add the vertices that no longer appear in $\underset{\sim}{s}'$ to the list below. Continue this until you are left with 2 vertices in the 2nd list: join these to complete T.

$\langle \cancel{2}, \cancel{4}, \cancel{1}, \cancel{2}, \cancel{5} \rangle$ $\qquad T =$ 

$\cancel{3}, \cancel{6}, 7, \cancel{4}, \cancel{1}, \cancel{2}, 5$

$2-3,\ 4-6,\ 1-4,\ 2-1,\ 5-2,$ plus $7-5$

4(b) List the vertices of G in increasing order as $d_1, d_2, \ldots, d_p$. Now suppose they are all distinct. Then $d_1 \geq 0$, $d_2 \geq 1$, $d_3 \geq 2, \ldots,$ $d_p \geq p-1$ because these are the smallest possible distinct degrees. But if $d_1 = 0$, then $d_p \leq p-2$ (because the maximum possible degree in G would be $p-2$) which contradicts $d_p \geq p-1$. And if $d_1 > 0$, then $d_1 \geq 1$ and this forces $d_2 \geq 2$, $d_3 \geq 3, \ldots, d_p \geq p$; which is again a contradiction because the maximum possible degree in G is $p-1$. So $d_1, \ldots, d_p$ can't be all distinct. Hence two vertices must have the same degree. This only works if $p \geq 2$.

6(b) By a theorem from the Homework Problems we know that

$$\text{No. of leaves in } T = 2 + \sum_{\deg(v_i) > 2} \{\deg(v_i) - 2\}$$
$$= 2 + 10(5-2) + 20(4-2) + 30(3-2)$$
$$= 2 + 30 + 40 + 30 = 102.$$

But I guess we still have to prove the Theorem. So this is half good.