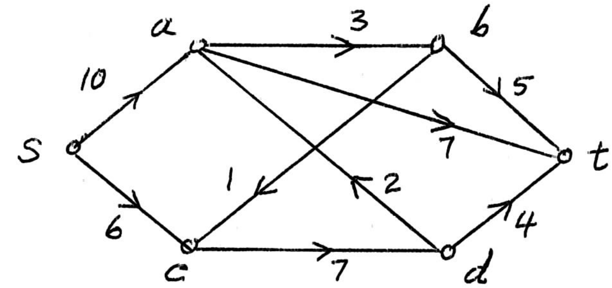
MAD 3305 - GRAPH THEORY FLORIDA INT'L UNIV.

TEST #2 - SPRING 2006 TIME: 75 min.

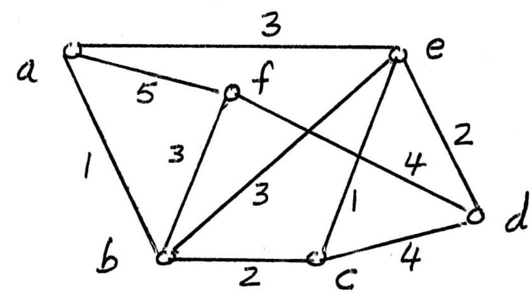
Answer all 6 questions. Provide all reasoning and show all working. An unjustified answer or failure to follow instructions will result in little credit. BEGIN EACH QUESTION ON A SEPARATE PAGE.

(15) 1. Find a maximal flow f^* in the network on the right by using the Ford-Fulkerson algorithm. Also find the source-separating set of vertices S^* corresp. to f^*.



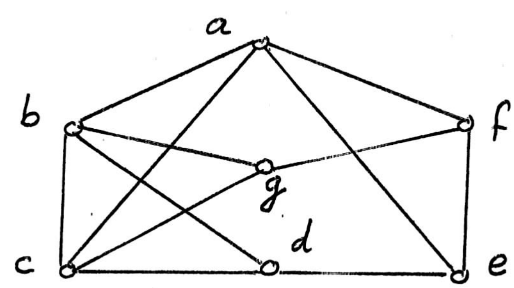
(15) 2. (a) Find a minimum postman walk of the graph on the right by using the Postman algorithm.

(b) What is the total length of your walk?



(20) 3. Using the DMP planarity algorithm, determine whether or not the graph on the right is planar.

(Show the embeddings for each step of the algorithm)



(15) 4. (a) Define what is a *legal flow* and what is the *value* of such a flow in a network N.
(b) Using the Euler-Circuit theorem, prove that a connected graph has an open Euler trail if and only if it has exactly two vertices of odd degree.

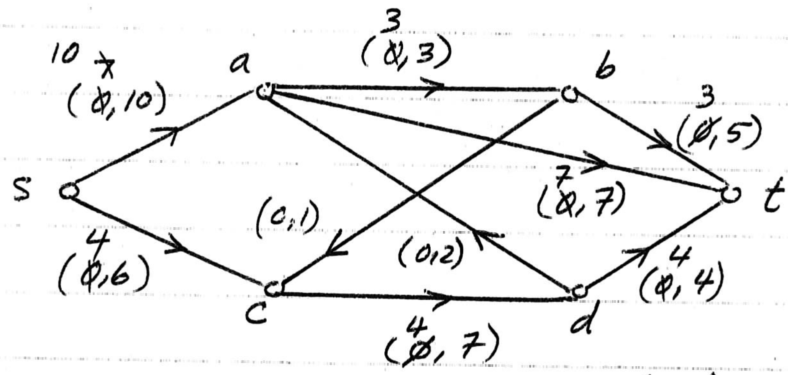
(15) 5. (a) Define what is a *minimum salesman walk* in a graph G.
(b) Prove that in any planar graph with k components we have $r=q+k+1-p$. (You may use *Euler's Formula*, if needed.)

(20) 6. (a) Define what is the *dual* of a planar graph G and what is a *self-dual* graph.
(b) Let G be a self-dual graph with p vertices and q edges. Prove that $q = 2.(p-1)$. Use this result to find a self-dual graph with 6 vertices.
[You may use any theorem proved in class for Qu.#6]

MAD 3305 - Graph Theory — Florida Int'l Univ.

Solutions to Test #2 — Spring 2006.

1.



1st Aug. semi-path: $s \xrightarrow{(0,10)} a \xrightarrow{(0,7)} t$

slacks: 10 7 $\lambda_1 = 7$

2nd Aug. semi-path: $s \xrightarrow{(7,10)} a \xrightarrow{(0,3)} b \xrightarrow{0,5} t$

slacks: 3 3 5 $\lambda_2 = 3$

3rd Aug. semi-path: $s \xrightarrow{(0,6)} c \xrightarrow{(0,7)} d \xrightarrow{(0,4)} t$

slacks 6 7 4 $\lambda_3 = 4$

$F(f^*)$ = net flow into t = $(3+7+4) - (0) = 14$

$S^* = \{v \in V(G)$: some flow can be sent from s to $v\}$

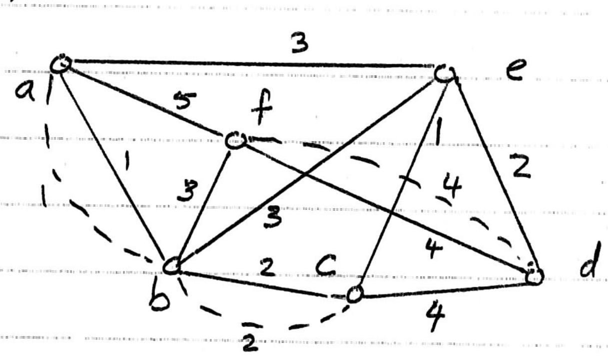
$= \{s, c, d, a\}$.

$c(S^*)$ = sum of outward capacities from S^*

$= 3 + 7 + 4 = 14 = F(f^*)$ ✓

2. Odd vertices are: a, c, d, f

dist.	a	c	d	f
a	.	3	5	4
c		.	3	5
d			.	4
f				



$\{a,c\} + \{d,f\}$ — $3+4=7$ ✓

$\{a,d\} + \{c,f\}$ — $5+5=10$

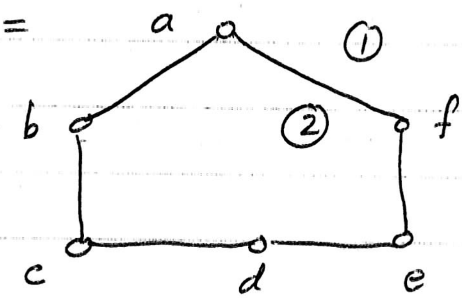
$\{a,f\} + \{c,d\}$ — $4+3=7$.

Min. Postman walk

$= a \xrightarrow{1} b \xrightarrow{2} c \xrightarrow{2} b \xrightarrow{1} a \xrightarrow{5} f \xrightarrow{4} d \xrightarrow{4} f \xrightarrow{3} b \xrightarrow{3} e \xrightarrow{1} c \xrightarrow{4} d \xrightarrow{2} e \xrightarrow{3} a$

Total length of the postman walk = 35.

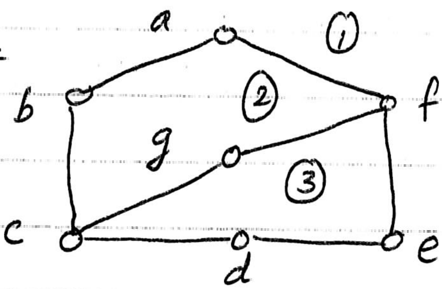
3. H_1 =

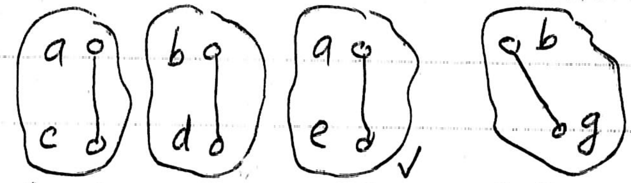


Segments relative to H_i

$\{1,2\}$ $\{1,2\}$ $\{1,2\}$ $\{1,2\}$

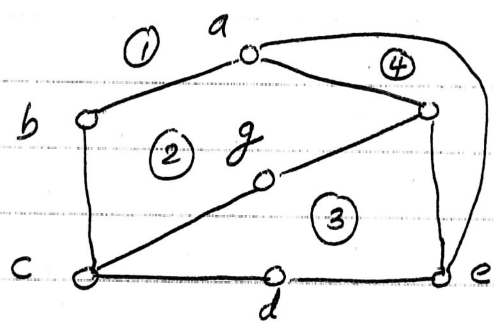
H_2 =

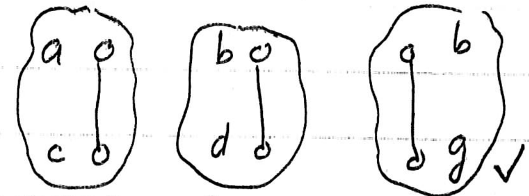




$\{1,2\}$ $\{1\}$ $\{1\}$ $\{2\}$

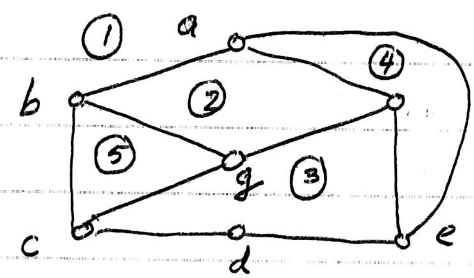
H_3 =

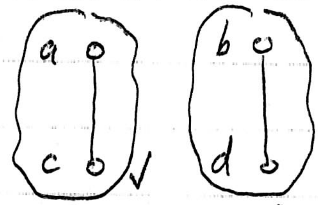




$\{1,2\}$ $\{1\}$ $\{2\}$

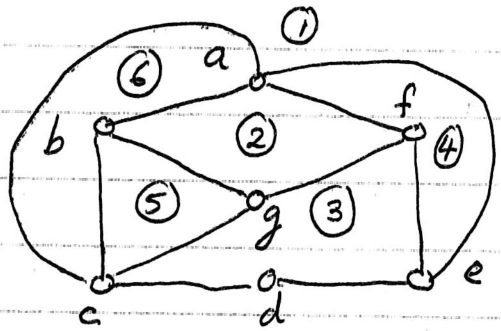
H_4 =

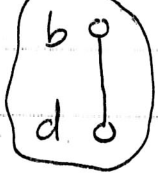




$\{1\}$ $\{1\}$

H_5 =





$\emptyset$ NON-PLANAR

G is non-planar because the segment b—d cannot be compatibly embedded in H_5. So G is non-planar

4(a) A legal flow in a network $N = \langle G, c, s, t\rangle$ is any function $f: E(G) \to [0,\infty)$ such that $f(e) \le c(e)$ for each $e \in E(G)$ and $\sum_{e \in In(v)} f(e) = \sum_{e \in Out} f(e)$ for each vertex $v \in V(G) - \{s,t\}$.

The value of the flow is defined by

$$F(f) = \sum_{e\in In(t)} f(e) - \sum_{e \in Out(t)} f(e).$$

(b) ($\Rightarrow$) Suppose G has an open Euler trail $v_1, v_2, \ldots, v_q$. Let $G' = G \cup \{v_q v_1\}$. Then $v_1, v_2, \ldots, v_q, v_1$ will be an Euler circuit of G'. So each vertex of G' must be of even degree because of the Euler circuit Theorem. Now $G = G' - \{v_q v_1\}$, so each vertex of G must have even degrees with the exception of v_q & v_1 which must have odd degrees. So G has exactly 2 vertices with odd degrees.

($\Leftarrow$) Suppose G has exactly 2 vertices with odd degrees. Let u and v be these vertices & put $G' = G \cup \{uv\}$. Then each vertex of G' will be of even degree. So G' will have an Euler circuit by the Euler circuit Theorem. Now if we remove the edge from this Euler circuit we will get an open Euler trail in G.

So G has an open Euler trail $\Leftrightarrow$ G has exactly 2 odd vertices.

5(a) A minimum salesman walk in a graph G is a closed walk of shortest possible length which includes each vertex of G at least once.

(b) Let $G_1, \ldots, G_k$ be the k components of G. Then

$$r(G_i) = q(G_i) + 2 - p(G_i) \quad \text{for each } i \in \{1, \ldots, k\}$$

So

$$\sum_{i=1}^{k} r(G_i) = \sum_{i=1}^{k} q(G_i) + \sum_{i=1}^{k} 2 - \sum_{i=1}^{k} p(G_i)$$

$$= q(G) + 2k - p(G).$$

5(b) But in the sum $\sum_{i=1}^{k} r(G_i)$, the infinite region has been counted k times (once for each G_i), so

$$r(G) = \left[\sum_{i=1}^{k} r(G_i)\right] - (k-1) = q(G) + 2k - p(G) - (k-1)$$

$$\therefore \quad r = q + (k+1) - p.$$

6(a) The dual of a planar graph G with respect to the embedding $\mathcal{E}$ of G is defined as follows: $V(G^*(\mathcal{E}))$ = the set of regions into which $\mathcal{E}$ partitions the plane and for each boundary that R_1 & R_2 share in $\mathcal{E}$, we will get an edge between R_1 & R_2 in $G^*(\mathcal{E})$.

A graph G is said to be self-dual if $G^*(\mathcal{E}) \cong G$ for any embedding $\mathcal{E}$ of G.

(b) Suppose G is a self-dual graph. Then $G^*(\mathcal{E}) \cong G$ for any embedding $\mathcal{E}$ of G. Now

$$r(\mathcal{E}) = r(G) = p(G^*(\mathcal{E})) \quad \text{by the definition of } G^*(\mathcal{E}).$$

And since $G^*(\mathcal{E}) \cong G$, $\quad p(G^*(\mathcal{E})) = p(G)$ because isomorphic graphs have the same number of vertices.

So $r(G) = p(G)$. Now we know that

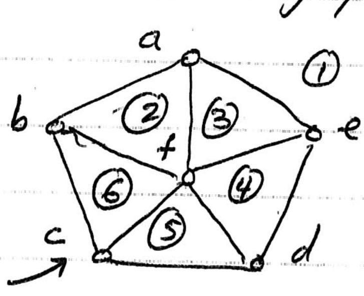
$$r(G) = q(G) + 2 - p(G)$$

by the Euler Planarity formula. So

$$p(G) = q(G) + 2 - p(G)$$

$$\therefore \quad 2p(G) - 2 = q(G). \qquad \therefore \quad q = 2(p-1).$$

From this formula, we know that a self-dual graph with 6 vertices will have $2(6-1) = 10$ edges. Such a graph is shown on the right



A self-dual graph with 6 vertices