MAD 3305 - GRAPH THEORY

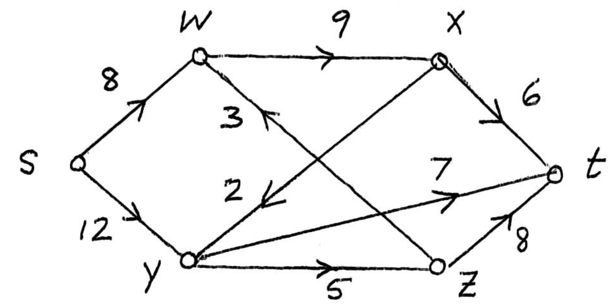
FLORIDA INT'L UNIV.

TEST #2 - SPRING 2007

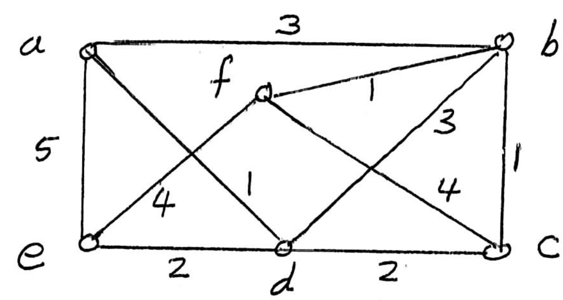
TIME: 75 min.

Answer all 6 questions. Provide all reasoning and show all working. An unjustified answer or failure to follow instructions will result in little credit. BEGIN EACH QUESTION ON A SEPARATE PAGE.

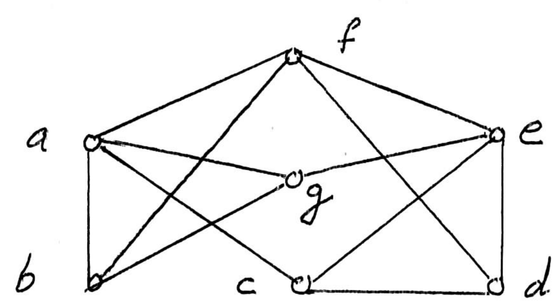
(15) 1. Find a maximal flow f^* in the network on the right by using the Ford-Fulkerson algorithm. Also find the source-separating set of vertices S^* corresp. to f^*.



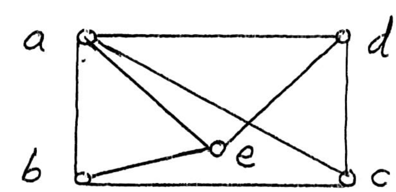
(15) 2. (a). Find a minimum postman walk of the graph on the right by using the Postman algorithm.
(b) What is the total length of your walk?



(15) 3. Determine whether or not the graph on the right is planar by using the DMP planarity algorithm. (Show the embeddings for each step of the algorithm)



(25) 4. (a) If a and b are non-adjacent vertices in G, prove that $P_G(\lambda) = P_{G\cup\{ab\}}(\lambda) + P_{G\circ\{ab\}}(\lambda)$.
(b) Find the Chromatic Polynomial of the graph G on the right.



(15) 5. (a) Define what are *Ore-type graphs* & *Euler circuits of G.*
(b) Suppose C is a cycle in an Ore-type graph G and x is a vertex in V(G)-V(C). Prove that we can find a path P in G which contains x and all the vertices of C.

(15) 6. (a) Define what are *simple polyhedra* & *polyhedral graphs.*
(b) Let H be a simple pòlyhedron with p vertices and q edges in which all faces have ≥ 6 edges. Prove that 2q ≤ 3p-6.
[You may use any theorem proved in class for Qu.#6]

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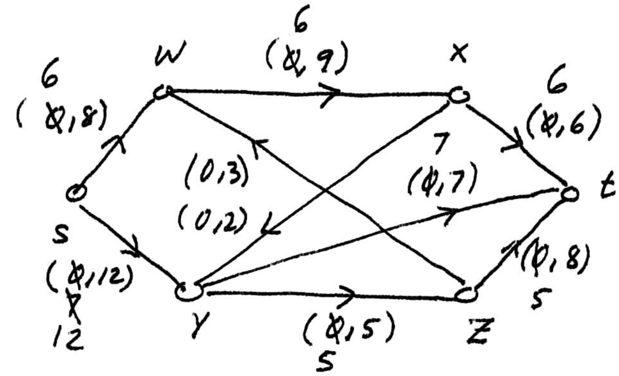
SOLUTIONS TO TEST # 2 SPRING 2007



1. 1st Aug. semi-path:

$s \xrightarrow{(0,12)} y \xrightarrow{(0,7)} t$

slacks: 12 , 7 ; $\lambda_1 = 7$

2nd Aug. semi-path:

$s \xrightarrow{(0,8)} w \xrightarrow{(0,9)} x \xrightarrow{(0,6)} t$

slacks: 8 , 9 , 6; $\lambda_2 = 6$

3rd Aug. semi-path: $s \xrightarrow{(7,12)} y \xrightarrow{(0,5)} z \xrightarrow{(0,8)} t$

slacks 5 , 5 , 8; $\lambda_3 = 5$

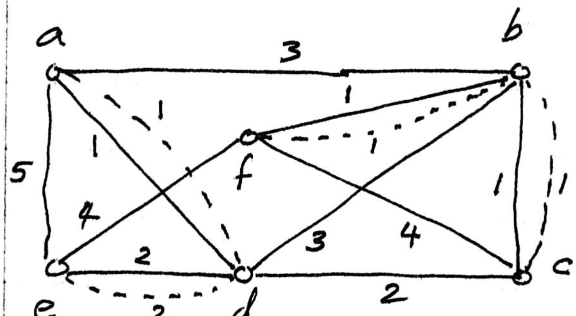
$S^* = \{u \in V(G)$: there is an aug. semi-path from s to u$\}$

$= \{s, w, x, y\}$, $c(S^*)$ = sum of outward capacities

$= 5 + 7 + 6 = 18$

$F(f^*)$ = net flow into $t = (6+7+5) - 0 = 18$ ✓

2(a) Odd vertices are: a, c, e, f



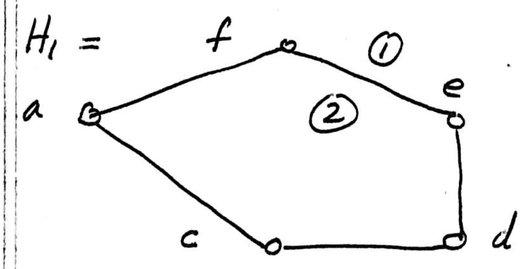
	a	c	e	f
a	·	3	3	4
c		·	4	2
e			·	4
f				·

{a,c}+{e,f}	{a,e}+{c,f}	{a,f}+{c,e}
3 + 4 = 7	3 + 2 = 5	4 + 4 = 8

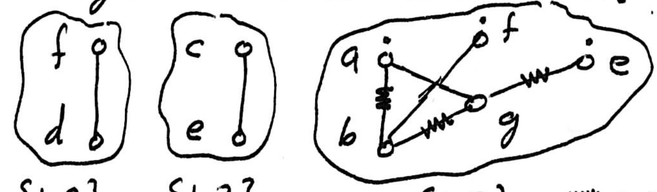
(b) A Minimum postman walk is: $a \xrightarrow{5} e \xrightarrow{2} d \xrightarrow{2} c \xrightarrow{1} b \xrightarrow{3} a \xrightarrow{1} d \xrightarrow{2} e \xrightarrow{4} f \xrightarrow{1} b \xrightarrow{1} f \xrightarrow{4} c \xrightarrow{1} b \xrightarrow{3} d \xrightarrow{1} a$

Length of minimum postman walk = 31.

3. $H_1 =$



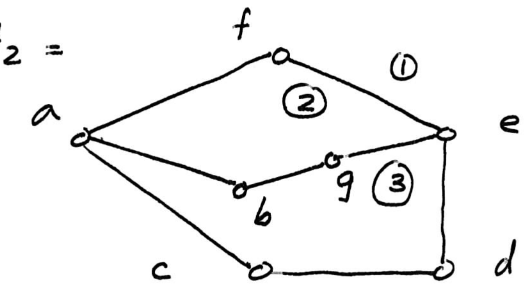
Segments of G w.r.t. H_i

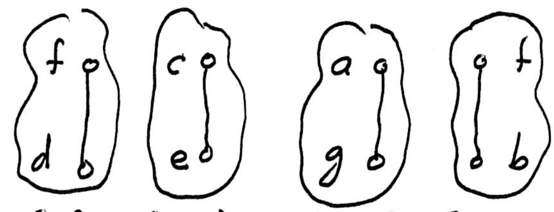


$\{1,2\}$ $\{1,2\}$ $\{1,2\}$

= path

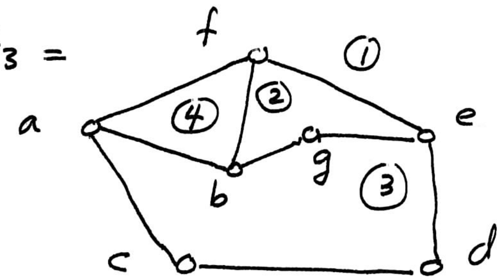
$H_2 =$

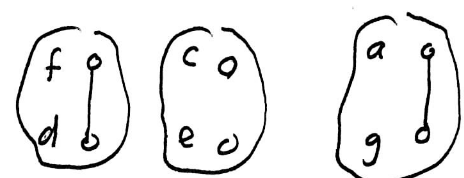




$\{1\}$ $\{1,3\}$ $\{2,3\}$ $\{2\}$

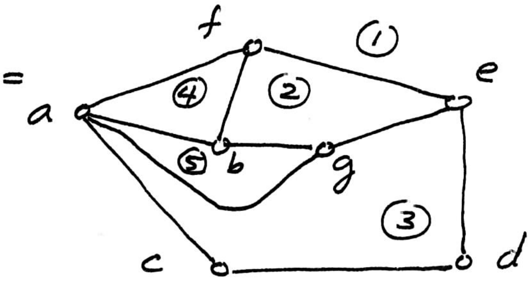
$H_3 =$

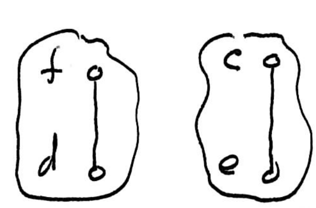




$\{1\}$ $\{1,3\}$ $\{3\}$

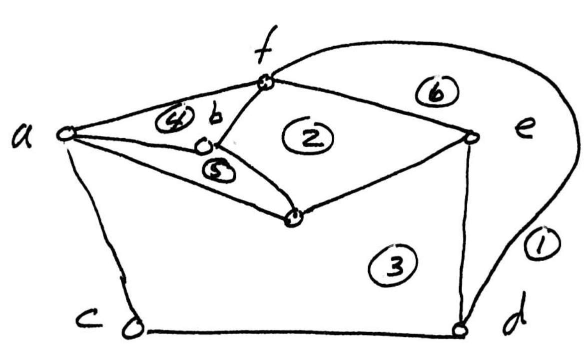
$H_4 =$

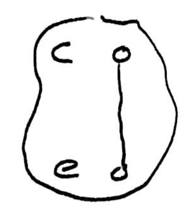




$\{1\}$ $\{1,3\}$

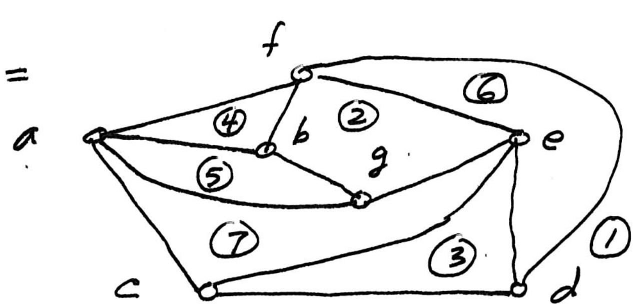
$H_5 =$





$\{3\}$

$H_6 =$



$\therefore$ G is planar.

4 (a) $P_G(\lambda)$ = Number of ways of legally coloring G with λ available colors

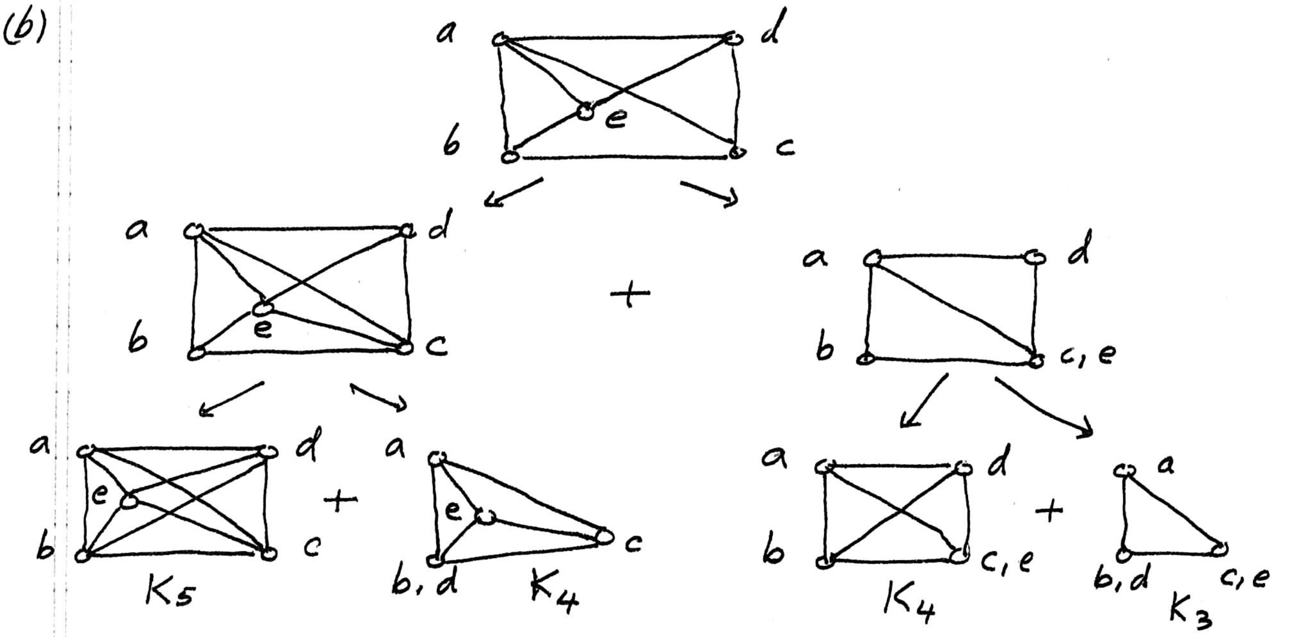
= No. of ways of legally coloring G with λ available colors with a & b receiving different colors

+

No. of ways of legally coloring G with λ available colors with a & b receiving the same colors

$= P_{G\cup\{ab\}}(\lambda) + P_{G\circ\{ab\}}(\lambda)$.

(b)



$\therefore\ P_G(\lambda) = P_{K_5}(\lambda) + P_{K_4}(\lambda) + P_{K_4}(\lambda) + P_{K_3}(\lambda)$

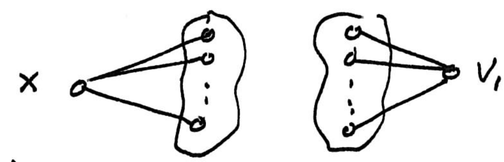
$= \lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4) + 2(\lambda)(\lambda-1)(\lambda-2)(\lambda-3) + \lambda(\lambda-1)(\lambda-2)$

$= \lambda(\lambda-1)(\lambda-2)[(\lambda-3)(\lambda-4) + 2(\lambda-3)+1] = \lambda(\lambda-1)(\lambda-2)(\lambda^2-5\lambda+7)$.

5(a) A graph G is said to be an Ore-type graph if for any pair of non-adjacent vertices x & y, $\deg(x)+\deg(y) \geq |V(G)|$.
An Euler circuit of G is a closed walk of G which uses each edge of G exactly once.

(b) There are two cases: Either x is adjacent to one of the vertices in $V(C)$ or x is adjacent to none of the vertices in $V(C)$.
Let the vertex sequence of the cycle be $v_1, v_2, v_3, \ldots, v_n, v_1$.

5 (b) Case (i) : x is adjacent to some vertex in $V(C)$, say v_i.
In this case $x, v_i, v_{i+1}, v_{i+2}, \cdots, v_n, v_1, v_2, \cdots, v_{i-2}, v_{i-1}$ is a path in G containing x and all of $V(C)$.
Case (ii) : x is adjacent to none of the v_i's.
In this case we claim that there is a vertex y such that xy and yv_1 are both in G. Suppose this is not so. Let A = set of vertices adjacent to x & B = set of vertices adjacent to v_1. Then $A \cap B = \emptyset$.
But $A \cup B \subseteq G - \{x, v_1\}$, so



$deg(x) + deg(v_1) \leq |V(G)| - 2 < |V(G)|$.
But x & v_1 are non-adjacent, so this contradicts the fact that G was Ore-type. Hence such a vertex y exists. Note $y \notin V(C)$ because we are in Case (ii). Now $x, y, v_1, v_2, \cdots, v_n$ is a path in G which contains x and all of $V(C)$.

6 (a) A simple polyhedron is a solid figure which is bounded by plane polygonal faces and which can be continuously distorted into a solid sphere. A polyhedral graph is any graph that can be obtained by considering the vertices & edges of a simple polyhedron as edges and vertices of a graph.
(b) Let $A_1, \ldots, A_r$ be the regions of a planar embedding of the polyhedral graph obtained from H. Then $e(A_i) \geq 6$ for each i and $e(A_1) + e(A_2) + \cdots + e(A_r) = 2q$ because each edge is in two regions. $\therefore$ $2q \geq 6 + 6 + 6 + \cdots + 6$ (r times)
$\therefore$ $2q \geq 6r$ and so $q \geq 3r$. But from Euler's formula we know that $r = q + 2 - p$. So

$$q \geq 3(q+2-p) \qquad \therefore \quad q \geq 3q + 6 - 3p$$

$$\therefore \quad 3p - 6 \geq 3q - q. \text{ Hence } \quad 2q \leq 3p - 6.$$