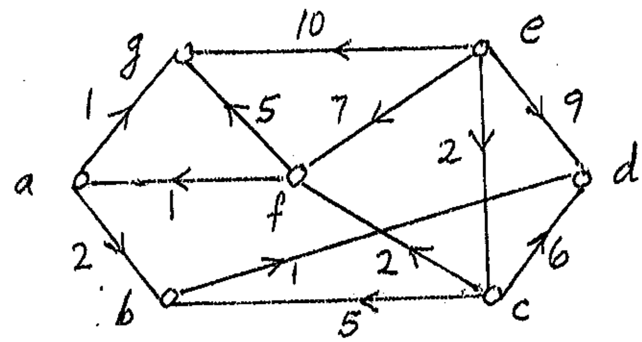
MAD 3305 - GRAPH THEORY FLORIDA INT'L UNIV.

TEST #1 - SPRING 2013 TIME: 75 min.

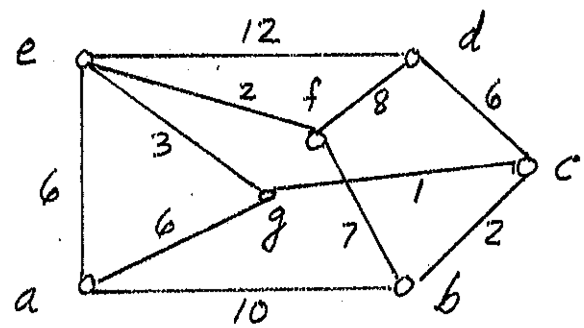
Answer *all 6* questions. **No calculators or Cellphones are allowed.** *An unjustified answer will receive little or no credit. BEGIN EACH OF THE 6 QUESTIONS ON A SEPARATE PAGE.*

(15) 1. Find the **distances** from **e** to each of the other vertices of the graph on the right by using *Dijkstra's Algorithm*.



(20) 2(a) Find a **graph** with degree sequence 4,3,3,2,2 by using the *Graphical Sequence Algorithm*.

(b) Find a **minimal spanning tree** of the graph on the right by using *Prim's Algorithm* & starting at **e**.



(20) 3(a) Find the **tree** corresponding to the sequence <1,5,2,1> via *Prufer's Tree Decoding Algorithm*.

(b) The five characters a,b,c,d,e occur with frequencies 5,3,2,1,9 respectively. Find an **optimal binary coding** for these five characters & the **W.P.L.** of your coding.

(15) 4(a) Define what is the ***adjacency matrix A*** of a graph G.

(b) Let G be a graph with vertices $v_1, \ldots, v_p$. Prove that the number of walks of length n from v_i to v_j is $(A^n)[i,j]$

(15) 5(a) Define what is the ***edge-connectivity*** of a connected graph.

(b) Let G be graph with p vertices and suppose that the minimum degree in G is $p/2$. Prove that G must be connected.

(15) 6(a) Define what is the ***height h(T)*** of a ***rooted tree T***.

(b) Let T be a binary tree with n vertices. Prove that

$$h(T) \geq \log_2 [(n+1)/2)].$$

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SOLUTIONS TO TEST #1 | SPRING 2013

1.

$L(a)$	$L(b)$	$L(c)$	$L(d)$	$L(e)$	$L(f)$	$L(g)$	T	V_0
∞	∞	∞	∞	0	∞	∞	$\{a,b,c,d,e,f,g\}$	e
∞	∞	2	9	.	7	10	$\{a,b,c,d,f,g\}$	c
∞	7	.	8	.	4	10	$\{a,b,d,f,g\}$	f
5	7	.	8	.	.	9	$\{a,b,d,g\}$	a
.	7	.	8	.	.	6	$\{b,d,g\}$	g
.	7	.	8	.	.	.	$\{b,d\}$	b
.	.	.	8	.	.	.	$\{d\}$	d
.	.	.	.	.	.	.	∅ STOP	
5	7	2	8	0	4	6	$= d(e,\cdot)$	

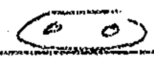
2(a)

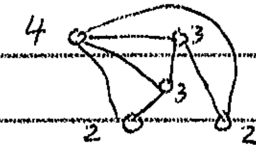
4, 3, 3, 2, 2 G_4

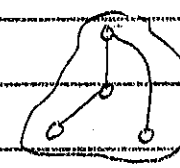
2, 2, 1, 1 G_3

1, 0, 1 $G_2 =$ 

1, 1, 0

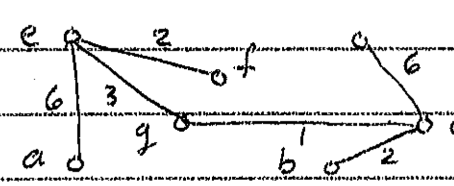
0, 0 $G_1 =$ 

$G_4 =$ 

$G_3 =$ 

(b)

$E(T)$	U	a	b	c	d	e	f	g	x_0
		$d(U,\cdot)$							
∅	$\{e\}$	6	∞	∞	12	.	2	3	f
$\{ef\}$	$\{e,f\}$	6	7	∞	8	.	.	3	g
$\{ef,eg\}$	$\{e,f,g\}$	6	7	1	8	.	.	.	c
$\{ef,eg,gc\}$	$\{e,f,g,c\}$	6	2	.	6	.	.	.	b
$\{ef,eg,gc,cb\}$	$\{e,f,g,c,b\}$	6	.	.	6	.	.	.	a
$\{ef,eg,gc,cb,ea\}$	$\{e,f,g,c,b,a\}$	.	.	.	6	.	.	.	d
$\{ef,eg,gc,cb,ea,cd\}$	$\{e,f,g,c,b,a,d\}$	STOP							

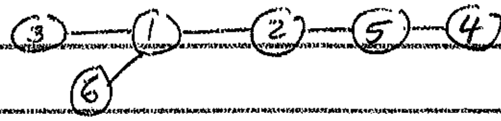
$T_{min} =$ 

3(a) $\underset{\sim}{s} = \langle 1,5,2,1 \rangle = \langle s_1, s_2, s_3, s_4 \rangle$. $\|\underset{\sim}{s}\| = 4 \Rightarrow p = \|\underset{\sim}{s}\| + 2 = 6$. Tree is on 6 vertices.

degrees that vertex x needs at stage i

i \ x	1	2	3	4	5	6	j — s_i (from $\underset{\sim}{s}$)
1	3	2	1	1	2	1	3 — 1
2	2	2	0	1	2	1	4 — 5
3	2	2	0	0	1	1	5 — 2
4	2	1	0	0	0	1	2 — 1
	1	0	0	0	0	1	plus 1 — 6

$\therefore T =$



(b)

d c b a e

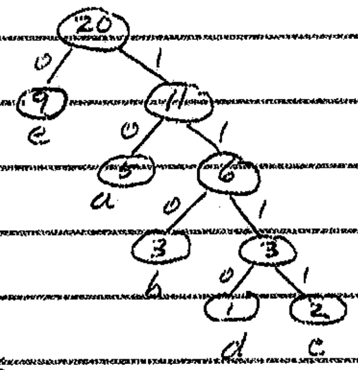
$\underbrace{1, 2}, 3, 5, 9$

$\underbrace{3, 3}, 5, 9$

$\underbrace{5, 6}, 9$

$\underbrace{9, 11}$

20



	c_1	c_2	c_3	c_4	c_5
Char	a	b	c	d	e
Code	10	110	1111	1110	0
freq. f_i	5	3	2	1	9
$f_i \cdot l_i$	10	9	8	4	9

$$\text{W.P.L.(coding)} = \sum_{i=1}^{k} f_i \cdot l_i = 10+9+8+4+9 = \boxed{40}$$

4(a) The adjacency matrix of a graph G on the vertices $v_1, \ldots, v_p$ is the $p \times p$ matrix A defined by $A[i,j]$ = no. of edges from v_i to v_j.

(b) We will prove the result by induction on n.

Basis: If $n=1$, then $A^1[i,j]$ = no. of edge from v_i to v_j = the no. of walks of length 1 from v_i to v_j. So the result is true for 1.

Ind. step. Suppose the result is true for n between any pair of vertices v_i & v_j. Then $A^n[i,j]$ = no. of walks of length n from v_i to v_j.

$$\text{So} \begin{pmatrix}\text{no. of walks of} \\ \text{length } n{+}1 \text{ from } v_i \text{ to } v_j\end{pmatrix} = \sum_{k=1}^{p} \begin{pmatrix}\text{no. of walks of} \\ \text{length } n \text{ from } v_i \text{ to } v_k\end{pmatrix} \cdot \begin{pmatrix}\text{no. of walks of} \\ \text{length 1 from } v_k \text{ to } v_j\end{pmatrix}$$

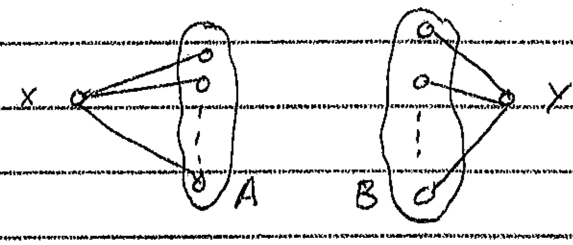
$$= \sum_{k=1}^{p} A^n[i,k] \cdot A^1[k,j] = A^{n+1}[i,j]$$

So if the result is true for n, it will be true for $n+1$.

Conclusion: By the principle of Mathematical Induction, it follows that the result is true for all values of n.

5(a) The edge-connectivity of a non-trivial graph is the smallest number of edges whose removal can disconnect the graph.

(b) Let G be a graph with p vertices & with minimum degree p/2. Suppose G is not connected. Then we can find two vertices x & y such that there is no path from x to y in G. Let $A = \{v \in V(G) : v \text{ is adjacent to } x\}$ and $B = \{v \in V(G) : v \text{ is adjacent to } y\}$ Then A & B are disjoint because there is no path from x to y in G. Also $|A \cup B| \leq p-2$ because neither x nor y can be in $A \cup B$.



Hence $|A| + |B| = |A \cup B| \leq p-2$. $\therefore$ $|A| \leq \frac{p-2}{2}$ or $|B| \leq \frac{p-2}{2}$ (because if $|A| > \frac{p-2}{2}$ & $|B| > \frac{p-2}{2}$, then $|A|+|B| > p-2$)

So $deg(x) = |A| \leq \frac{p-2}{2} < \frac{p}{2}$ or $deg(y) = |B| \leq \frac{p-2}{2} < \frac{p}{2}$. But this contradicts the fact that the minimum degree in G was p/2. Hence G must be connected.

6(a) The height of a rooted tree is the highest level that exists in the tree. (The vertices of the rooted tree are classified into levels according to their distances from the root).

(b) First observe that there is only one vertex at level 0. Since we have a binary tree, there will be at most 2^1 vertices at level 1, at most 2^2 vertices at level 2, and so on. In general there will be at most 2^i vertices at level i.

Let $k = h(T)$. Then

$$n \leq 1 + 2^1 + 2^2 + \cdots + 2^k$$
$$= (2^{k+1}-1)/(2-1) = 2^{k+1} - 1$$

$\therefore$ $n+1 \leq 2^{k+1}$ $\therefore$ $\frac{n+1}{2} \leq 2^k$

$\therefore$ $\log_2(2^k) \geq \log_2\left(\frac{n+1}{2}\right)$. $\therefore$ $k = h(T) \geq \log_2\left(\frac{n+1}{2}\right)$.