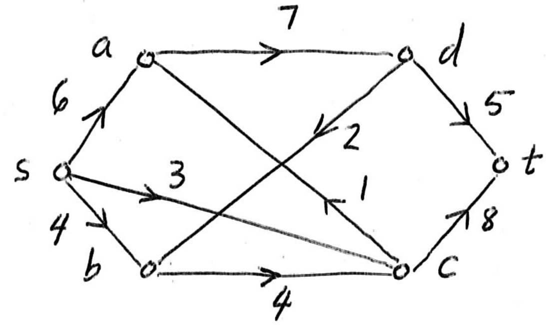
MAD 3305 - GRAPH THEORY FLORIDA INT'L UNIV.

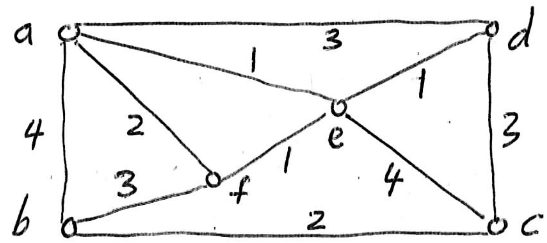
TEST #2 - SPRING 2013 TIME: 75 min.

Answer all 6 questions. Provide all reasoning and show all working. An unjustified answer will receive little or no credit. BEGIN EACH QUESTION ON A SEPARATE PAGE.

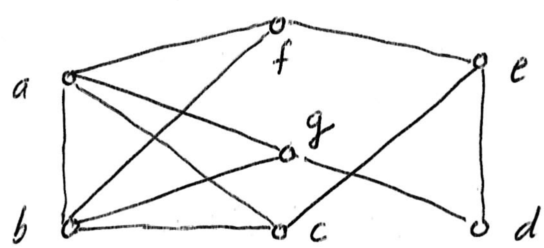
(15) 1. Find a maximal flow f^* in the network on the right by using the Ford-Fulkerson algorithm. Also find the source-separating set of vertices S^* corresponding to f^*.



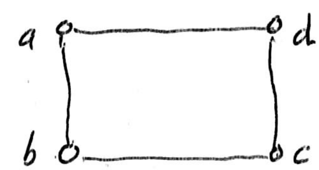
(15) 2(a) Find a minimum postman walk of the graph on the right by using the Postman algorithm.
(b) What is the total length of your walk?



(18) 3. Determine whether or not the graph on the right is planar by using the DMP planarity algorithm. [*Show the embeddings for each step of the algorithm.*]



(22) 4(a) Find the Chromatic Polynomial of the graph G on the right.
(b) Prove that the chromatic number of a non-trivial T tree is 2.

$G =$ 

(15) 5(a) Define what is a ***minimum salesman walk*** of a graph *G*.
(b) Use the *Euler Circuit Theorem* to prove that a connected graph has an open Euler trail if and only if it has exactly two vertices of odd degree.

(15) 6(a) Define what is the ***dual*** of a planar graph G with respect to the planar embedding E.
(b) Let p_i = number of vertices of degree i in a maximal planar graph G with $\geq$ 4 vertices. Prove that $3P_3 + 2p_4 + p_5 \geq 12$.

[*You may use any theorem proved in class for Qu.#6*]

MAD 3305 - Graph Theory | Florida Internat'l Univ.

Solutions to Test #2 | Spring 2013

1(a) 1st aug. semi-path:

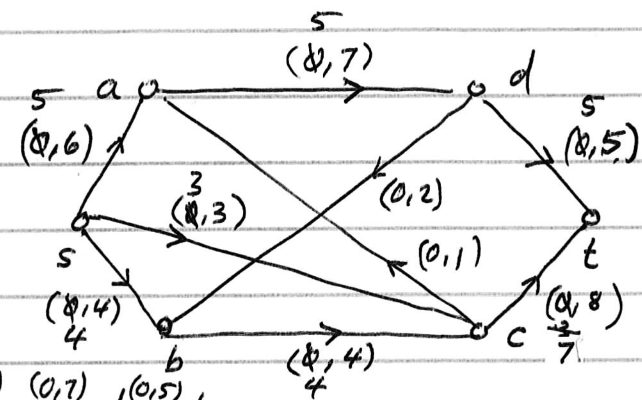
$s \xrightarrow{(0,3)} c \xrightarrow{(0,8)} t$

slacks: 3, 8; $\lambda_1 = 3$

2nd aug. semi-path:

$s \xrightarrow{0,4} b \xrightarrow{0,4} c \xrightarrow{(3,8)} t$

slacks: 4, 4, 5; $\lambda_2 = 4$



3rd aug. semi-path: $s \xrightarrow{(0,6)} a \xrightarrow{(0,7)} d \xrightarrow{(0,5)} t$

slacks: 6, 7, 5; $\lambda_3 = 5$.

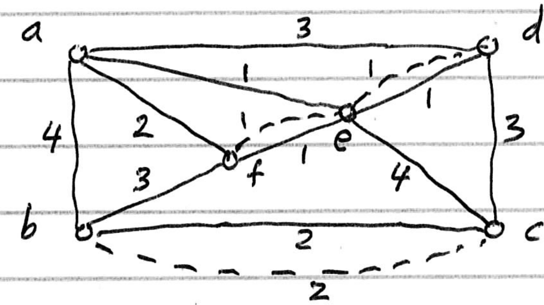
(b) $S^* = \{u \in V(G)$: there is an aug. semi-path from s to $u\} = \{s, a, d, b\}$.

$c(S^*)$ = sum of outward capacities $= 4+3+5 = 12$.

$F(f^*)$ = net flow into $t = (7+5) - (0) = 12$. ✓

2(a) Odd vertices: $\{b, c, d, f\}$

dist.	b	c	d	f
b	.	2	5	3
c		.	3	5
d			.	2
f				.



$\{b,c\} + \{d,f\}$; $2+2=4$ ✓

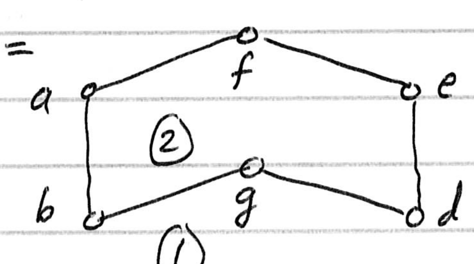
$\{b,d\} + \{c,f\}$; $5+5=10$

$\{b,f\} + \{c,d\}$; $3+3=6$

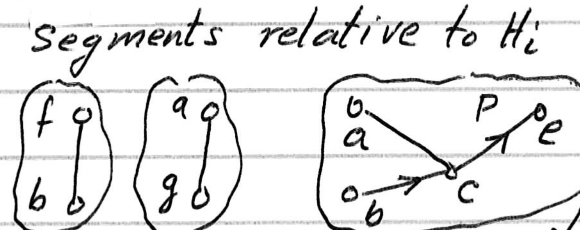
Minimum Postman walk is: $a \xrightarrow{4} b \xrightarrow{2} c \xrightarrow{2} b \xrightarrow{3} f \xrightarrow{1} e \xrightarrow{1} d \xrightarrow{1} e \xrightarrow{1} f \xrightarrow{2} a \xrightarrow{1} e \xrightarrow{4} c \xrightarrow{3} d \xrightarrow{3} a$.

(b) Length $= 4+2+2+3+1+1+1+1+2+1+4+3+3 = 28$.

3. $H_1 =$



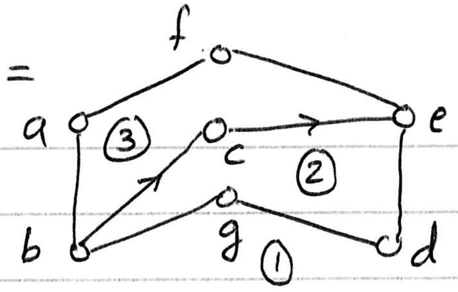
Segments relative to H_1



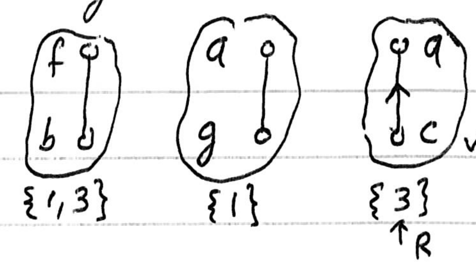
$\{1,2\}$ $\{1,2\}$ $\{1,2\}$ ✓

↑ R

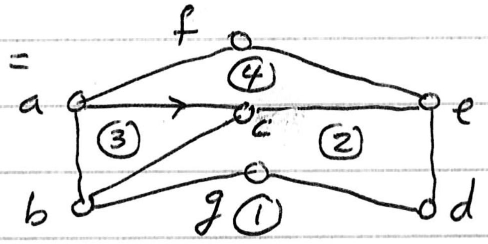
3. $H_2 =$

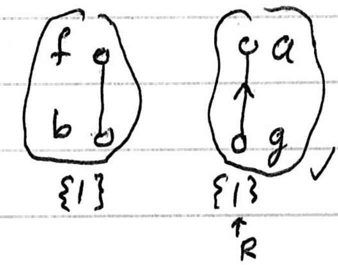


Segments relative to H_i

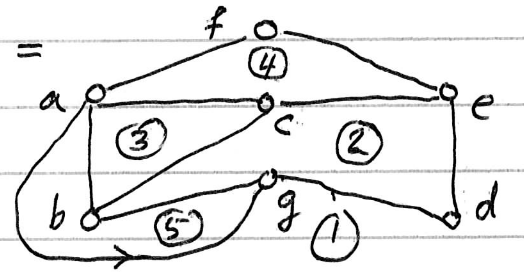


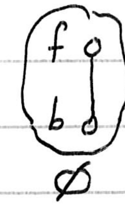
$H_3 =$





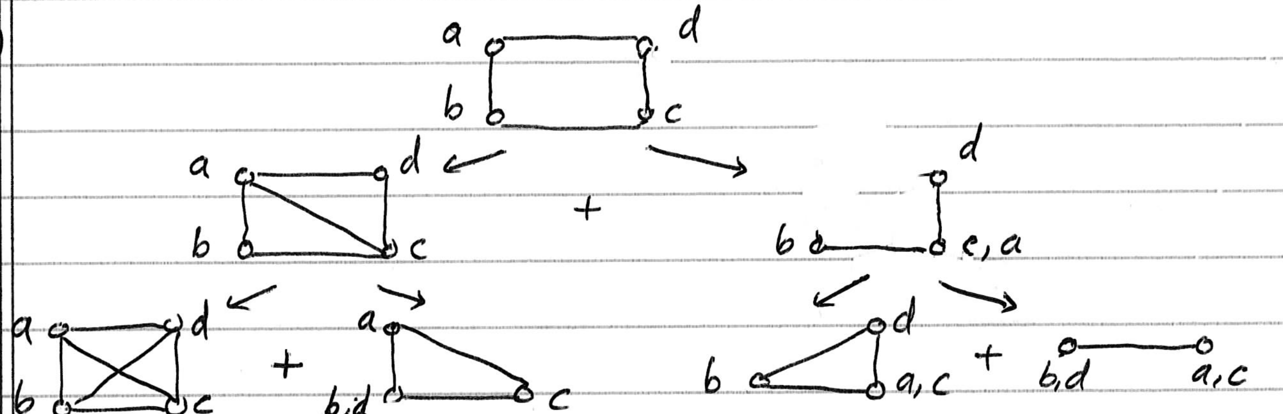
$H_4 =$





$\therefore$ G is NON-PLANAR

4(a)



$\therefore P_G(\lambda) = P_{K_4}(\lambda) + 2P_{K_3}(\lambda) + P_{K_2}(\lambda) = \lambda(\lambda-1)(\lambda-2)(\lambda-3) +$

$2\lambda(\lambda-1)(\lambda-2) + \lambda(\lambda-1) = \lambda(\lambda-1)[(\lambda-2)(\lambda-3) + 2(\lambda-2) + 1]$

$= \lambda(\lambda-1)[\lambda^2 - 3\lambda + 3]$

(b) Since T is a non-trivial tree, T has at least 1 edge and this edge alone needs 2 colors. So $\chi(T) \geq 2$. Now select any vertex v_0 in T and designate it as the root of T. Then the vertices of T will be split into levels according to their distances from v_0. Color the vertices in the even levels with color #1 & the vertices at the odd levels with color #2. This will give us a legal coloring of T because vertices at even levels can't be adjacent. The same is true for odd levels. $\therefore \chi(T) = 2$.

5(a) A minimum salesman walk is a closed walk of shortest total length which includes all the vertices of G.

(b) Suppose G has an open Euler trail $v_1, v_2, \ldots, v_q$. Let $G' = G \cup \{v_q v_1\}$. Then $v_1, v_2, \ldots, v_q, v_1$ will be an Euler circuit of G'. So each vertex of G' must be of even degree because of the Euler Circuit Theorem. But $G = G' - \{v_q v_1\}$. So each vertex of G must be of even degree except for v_q & v_1 which will be of odd degree. Hence G has exactly 2 vertices of odd degree.

Now suppose G has exactly 2 vertices, u & v say, of odd degree. Put $G' = G \cup \{uv\}$. Then each vertex of G' will be of even degree. So G' will have an Euler circuit Q by the Euler Circuit Theorem. Now if we remove uv from Q we will get an open Euler trail of G.

$\therefore$ G has an open Euler trail $\Leftrightarrow$ G has exactly 2 vertices of odd degree.

6(a) $V(G^*(E))$ = set of regions into which E partitions the plane; & for each boundary that R_1&R_2 share in E, we get an edge between the regions R_1&R_2 in $G^*(E)$.

(b) Let k be the maximum possible degree in G. Since G is maximal planar, G cannot have any vertices of degree 0, 1 or 2. So $2q$ = sum of degrees in G

$$= 3 \cdot p_3 + 4 \cdot p_4 + 5 p_5 + \cdots + k \cdot p_k$$

But G is maximal planar, so $q = 3p - 6$ by a Theorem in class.

$$\therefore \quad 2(3p-6) = 3 \cdot p_3 + 4 \cdot p_4 + 5 \cdot p_5 + \cdots + k \cdot p_k.$$

Also $p = p_3 + p_4 + \cdots + p_k$. Hence we have

$$6(p_3 + p_4 + \cdots + p_k) - 12 = 3p_3 + 4p_4 + \cdots + k \cdot p_k$$

$$\therefore (6-3)p_3 + (6-4)p_4 + (6-5)p_5 = 12 + (6-6)p_6 + (7-6)p_7 + \cdots + (k-6)p_k$$

$$\therefore 3 \cdot p_3 + 2 \cdot p_4 + 1 \cdot p_5 = 12 + \underbrace{1 \cdot p_7 + 2 \cdot p_8 + \cdots + (k-6)p_k}_{\geq 0}.$$

$$\therefore 3p_3 + 2p_4 + p_5 \geq 12.$$

END.