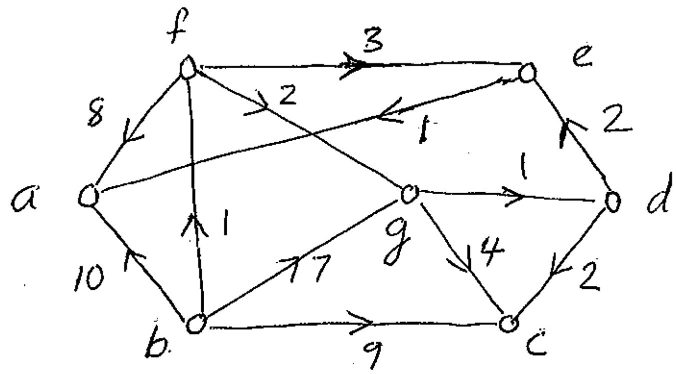
MAD 3305 - GRAPH THEORY FLORIDA INT'L UNIV.

TEST #1 - SPRING 2014 TIME: 75 min.

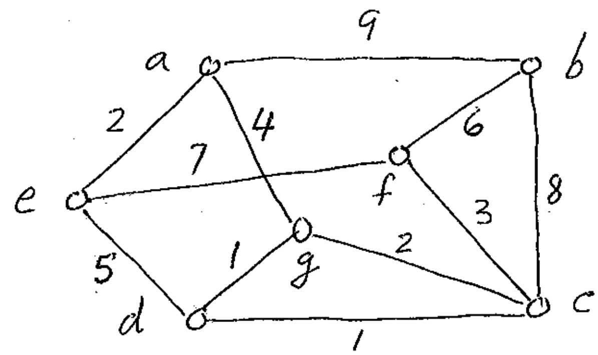
Answer *all 6* questions. **No calculators or Cellphones are allowed.** *An unjustified answer will receive little or no credit. BEGIN EACH OF THE 6 QUESTIONS ON A SEPARATE PAGE.*

(15) 1. Find the **distances** from ***b*** to each of the other vertices of the graph on the right by using *Dijkstra's Algorithm.*



(20) 2(a) Find a **graph** with degree sequence 4,3,3,3,1 by using the *Graphical Sequence Algorithm.*

(b) Find a **minimal spanning tree** of the graph on the right by using *Kruskal's Algorithm* (show partitions).



(20) 3(a) Find the **tree** corresponding to the sequence <1,3,1,4> via *Prufer's Tree Decoding Algorithm.*

(b) The five characters a,b,c,d,e occur with frequencies 1,2,3,5,8 respectively. Find an **optimal binary coding** for these five characters & the **W.P.L.** of your coding.

(15) 4(a) Define what is the ***vertex-connectivity*** $k_v(G)$ of a graph G.

(b) Let T be a tree. Prove that $|E(T)| = |V(T)| - 1$.

(15) 5(a) Define what is the ***distance from u to v*** in a graph G.

(b) Let G be disconnected graph. Prove $d(u,v) \leq 2$ for any pair of vertices in the compliment G^c of G.

(15) 6(a) Define what is the ***height h(T)*** of a ***rooted tree T.***

(b) Let T be a ternary (3-ary) tree with n vertices. Prove that $h(T) \geq \log_3 [(2n+1)/3)]$.

MAD 3305 - Graph Theory — Florida International Univ.

Solutions to Test #1 — Spring 2014

1.

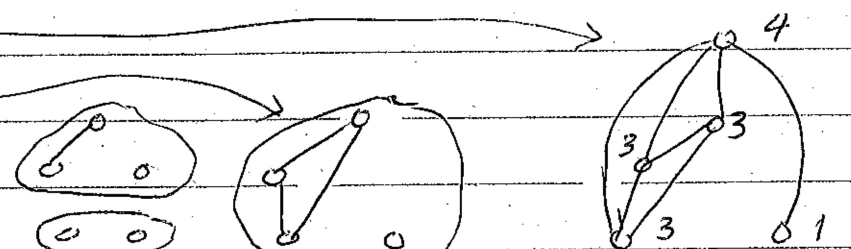
$L(a)$	$L(b)$	$L(c)$	$L(d)$	$L(e)$	$L(f)$	$L(g)$	T	i	v_0
∞	$\underline{0}$	∞	∞	∞	∞	∞	$\{a,b,c,d,e,f,g\}$	0	b
10	,	9	∞	∞	$\underline{1}$	7	$\{a,c,d,e,f,g\}$	1	f
9	,	9	∞	4	,	$\underline{3}$	$\{a,c,d,e,g\}$	2	g
9	,	7	$\underline{4}$	4	,	,	$\{a,c,d,e\}$	3	d
9	,	6	,	$\underline{4}$	,	,	$\{a,c,e\}$	4	e
$\underline{5}$	,	6	,	,	,	,	$\{a,c\}$	5	a
,	,	$\underline{6}$	,	,	,	,	$\{a\}$	6	c
,	,	,	,	,	,	,	$\emptyset$ STOP		
Ans 5	0	6	4	4	1	3	$= d(b,\cdot)$		

2 (a) $4, \underline{3,3,3,1}$

$2, \underline{2,2},0$

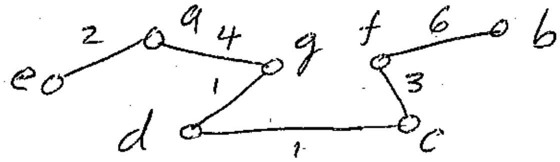
$1,\underline{1},0 \longrightarrow$

$0,0 \longrightarrow$



(b) $e_1 = cd$, $e_2 = dg$, $e_3 = ae$, $e_4 = cg$, $e_5 = cf$, $e_6 = ag$, $e_7 = de$, $e_8 = bf$, $e_9 = ef$, $e_{10} = bc$, $e_{11} = ab$.

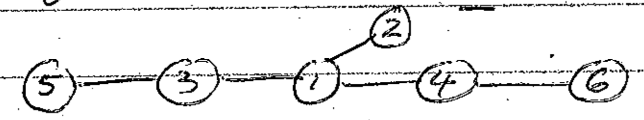
$E(T)$	Parts of Partition P	i	endpts(e_i)
$\emptyset$	$\{a\}\{b\}\{c\}\{d\}\{e\}\{f\}\{g\}$	1	$\{c,d\}$
$\{cd\}$	$\{a\}\{b\}\{c,d\}\{e\}\{f\}\{g\}$	2	$\{d,g\}$
$\{cd, dg\}$	$\{a\}\{b\}\{c,d,g\}\{e\}\{f\}$	3	$\{a,e\}$
$\{cd, dg, ae\}$	$\{a,e\}\{b\}\{c,d,g\}\{f\}$	4	$\{c,g\}$
don't add cg	$\{a,e\}\{b\}\{c,d,g\}\{f\}$	5	$\{c,f\}$
$\{cd, dg, ae, cf\}$	$\{a,e\}\{b\}\{c,d,f,g\}$	6	$\{a,g\}$
$\{cd, dg, ae, cf, ag\}$	$\{a,c,d,e,f,g\}\ \{b\}$	7	$\{d,e\}$
don't add de	$\{a,c,d,e,f,g\}\ \{b\}$	8	$\{b,f\}$
$\{cd, dg, ae, cf, ag, bf\}$	$\{a,b,c,d,e,f,g\}$	STOP, because P has 1 part	

2(b) Min. spanning tree = 

$|\underset{\sim}{S}| = 4 \Rightarrow p = 4+2 = 6.$

3(a)

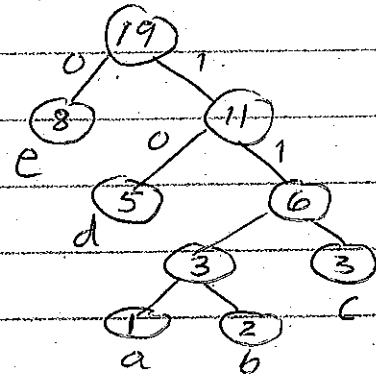
$d_i(1)$	$d_i(2)$	$d_i(3)$	$d_i(4)$	$d_i(5)$	$d_i(6)$	i	$l(i)$ — $s(i)$
$\underline{3}$	$\underline{1}$	2	2	1	1	1	2 — 1
2	0	$\underline{2}$	2	$\underline{1}$	1	2	5 — 3
$\underline{2}$	0	$\underline{1}$	2	0	1	3	3 — 1
$\underline{1}$	0	0	$\underline{2}$	0	1	4	1 — 4
0	0	0	$\underline{1}$	0	$\underline{1}$	5	4 — 6

T = 

(b)

Char.	a	b	c	d	e
Freq.	1	2	3	5	8
Code	1100	1101	111	10	0
length	4	4	3	2	1

1, 2, 3, 5, 8
3, 3, 5, 8
5, 6, 8
8, 11
19



$W.P.L = 4(1) + 4(2) + 3(3) + 5(2) + 8(1)$
$= 4 + 8 + 9 + 10 + 8 = 39.$

4 (a) $k_v(G)$ = smallest no. of vertices whose removal can disconnect G or reduce it to K_1.

(b) We will prove the result by induction on $|V(T)| = p$. For $p = 1$, $T = K_1$ & $E(T) = \emptyset$. So $|E(T)| = 0 = 1-1 = |V(T)| - 1$. So the result is true for $p=1$. Now suppose the result is true for all trees with $\leq p$ vertices. Let T be any tree with $p+1$ vertices. Choose any edge e in T. Then $T - \{e\}$ will consist of two trees T_1 & T_2. So

$|E(T)| = |E(T_1)| + |E(T_2)| + |\{e\}| = |V(T_1)| - 1 + |V(T_2)| - 1 + 1$
$= |V(T_1)| + |V(T_2)| - 1 = |V(T)| - 1.$ Hence if the resul. is true for all trees with $\leq p$ vertices, it will be true for all trees with $p+1$ vertices. Hence the result is true for all trees by Math Ind

5(a) $d(u,v) = \begin{cases} \text{length of the shortest path from } u \text{ to } v \text{ in } G \\ +\infty, \text{ if there is no path from } u \text{ to } v \text{ in } G. \end{cases}$

(b) Let u and v be any vertices in G^c

Case (i): $u = v$. In this case $d(u,v) = d(u,u) = 0$

Case (ii) $u \neq v$ & $uv \in E(G)$.

In this case, u & v will be in the same component of G. Let w be a vertex from another component of G (not containg u & v). Then uw & $wv \in E(G^c)$. So $d(u,v) = 2$.

Case (iii) $u \neq v$ & $uv \notin E(G)$

In this case, $uv \in E(G^c)$. So $d(u,v) = 1$ because u, v is the shortest path from u to v in G^c.

So in all three cases $d(u,v) \leq 2$.

6 (a) $h(T)$ = highest level that exists in the rooted tree T. (The vertices of the rooted tree are classified into levels according to their distances from the root of T.)

(b) Let $k = h(T)$. Since T is a ternary tree, level 0 will contain at most $1 = 3^0$ vertices; level 1 will contain at most 3^1 vertices (because each vertex at level 0 can have at most 3 children); level 2 will contain at most 3^2 vertices. In general level i will contain at most 3^i vertices. So

$$1 + 3^1 + 3^2 + \cdots + 3^k \geq \text{no. of vertices in } T.$$

$$\therefore \quad (3^{k+1} - 1)/(3-1) \geq n.$$

Thus $\quad 3^{k+1} - 1 \geq 2n \Rightarrow 3^{k+1} \geq 2n+1$

$\therefore \quad 3^k \geq (2n+1)/3. \quad \therefore \quad k \geq \log_3\left(\frac{2n+1}{3}\right)$

Thus $\quad h(T) \geq \log_3\left(\frac{2n+1}{3}\right)$.