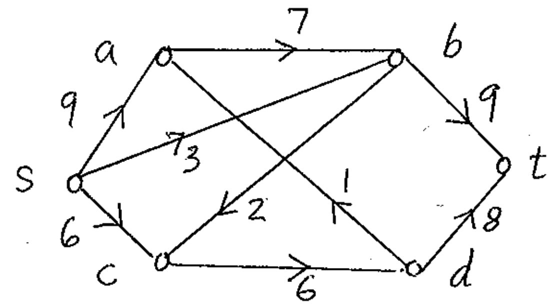
<u>MAD 3305 - GRAPH THEORY</u> <u>FLORIDA INT'L UNIV.</u>

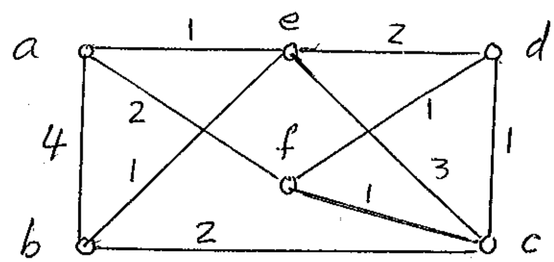
<u>TEST #2 - SPRING 2014</u> <u>TIME: 75 min.</u>

Answer all 6 questions. Provide all reasoning and show all working. An unjustified answer will receive little or no credit. BEGIN EACH QUESTION ON A SEPARATE PAGE.

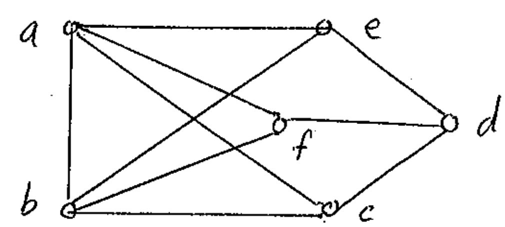
(15) 1. Find a ***maximal flow*** f^* in the network on the right by using the Ford-Fulkerson algorithm. Also find the ***source-separating set*** of vertices S^* corresponding to f^*.



(15) 2. Find a ***minimum postman walk*** of the graph on the right by using the ***Postman algorithm***; and the ***total length*** of your minimum postman walk?

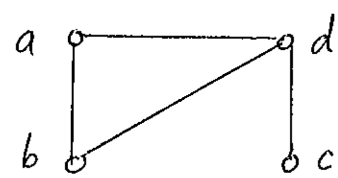


(18) 3. Determine whether or not the graph on the right is planar by using the ***DMP Planarity algorithm***. [*Show the embeddings for each step of the algorithm.*]



(22) 4(a) Find $P_G(\lambda)$ for the graph G on the right by using the ***Chromatic Polynomial algorithm***.

(b) Prove that $P_T(\lambda) = \lambda(\lambda-1)^{n-1}$ for any tree T with n vertices.



(15) 5(a) Define what is a ***minimum salesman walk*** in a graph G.

(b) Use ***Ore's Theorem*** to prove that if G is a graph with p vertices and $deg(x)+deg(y) \geq p-1$ for any pair of non-adjacent vertices x & y in G, then G has a ***Hamilton path***.

(15) 6(a) Define what is a ***simple polyhedron***.

(b) Let G be a simple polyhedron with no triangular faces. Prove that $q \leq 2p-4$.

[*You may use any theorem proved in class for Qu.#6*]

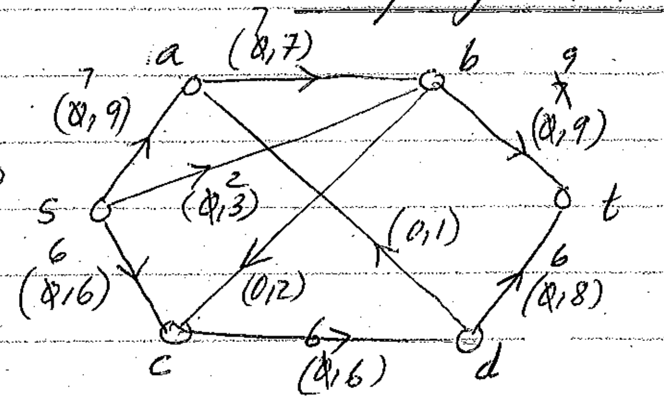
MAD 3305 - Graph Theory Florida International Univ.

Solutions to Test #2 Spring 2014

1 (a) 1st augmenting semi-path:

$s \xrightarrow{(0,9)} a \xrightarrow{(0,7)} b \xrightarrow{(0,9)} t$

Slacks: 9 7 9 $\mu_1 = 7$



2nd augmenting semi-path:

$s \xrightarrow{(0,3)} b \xrightarrow{(7,9)} t$

Slacks: 3 2 $\mu_2 = 2$

3rd augmenting semi-path: $s \xrightarrow{(0,6)} c \xrightarrow{(0,6)} d \xrightarrow{(0,8)} t$

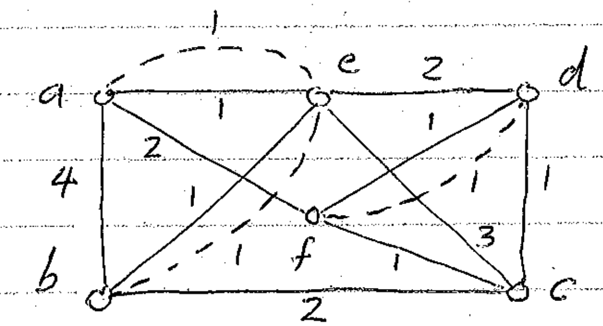
slacks: 6 6 8 $\mu_3 = 6$

$Val(f^*) = \text{net flow into } t = f^*(\vec{bt}) + f^*(\vec{dt}) = 9 + 6 = 15$

$S^* = \{u \in V(G) : \text{there is an aug. semi-path from } s \text{ to } u\} = \{s, a, b, c\}$

$c(S^*) = \text{sum of outward capacities} = c(\vec{cd}) + c(\vec{bt}) = 6 + 9 = 15 \checkmark$

2 (a) Odd vertices: $\{a, b, d, f\}$



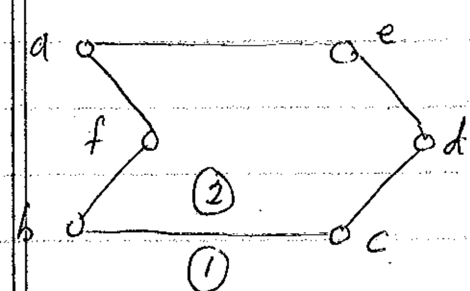
	a	b	d	f
a	.	$\boxed{2}$	3	2
b		.	3	3
d			.	$\boxed{1}$
f				.

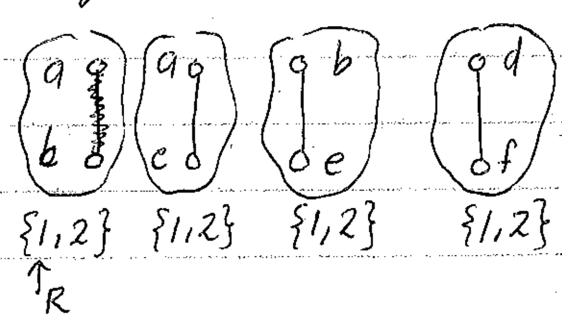
$\{a,d\} + \{d,f\}$ $\{a,d\} + \{b,f\}$ $\{a,f\} + \{b,d\}$

$2 + 1$ $3 + 3$ $2 + 3$

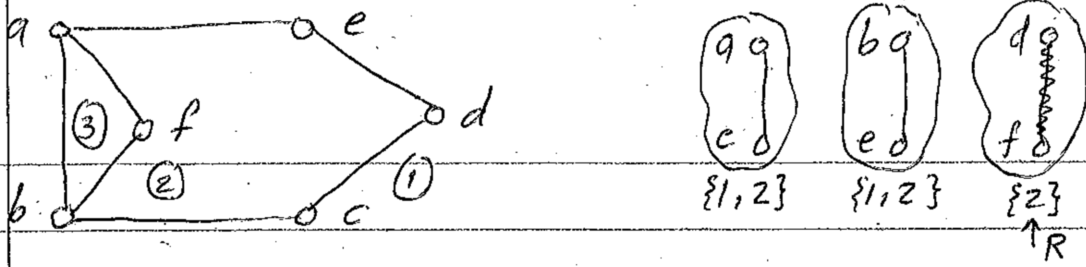
Minimum Postman walk $= a \overset{4}{-} b \overset{2}{-} c \overset{1}{-} d \overset{1}{-} f \overset{1}{-} d \overset{2}{-} e \overset{1}{-} a \overset{1}{-} e \overset{1}{-} b \overset{1}{-} e \overset{3}{-} c \overset{1}{-} f \overset{2}{-} a$. (b) Total length $= 21 = 18 + 3 \checkmark$

3. Embedding of H_i Segments of G relative to H_i

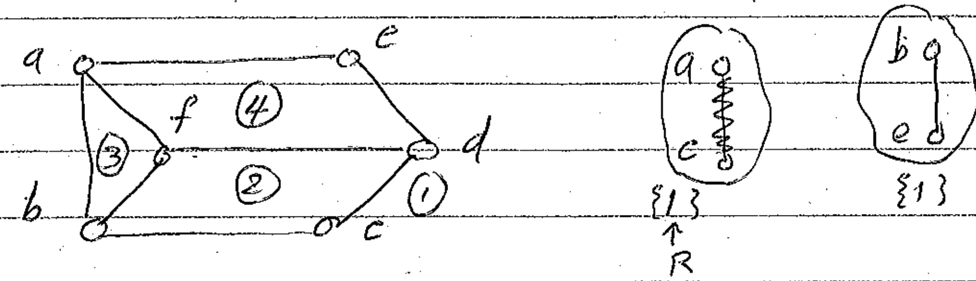




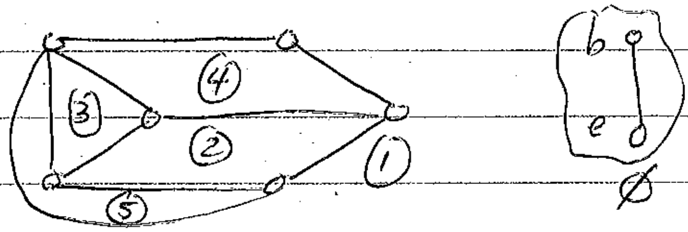
$H_2 =$



$H_3 =$



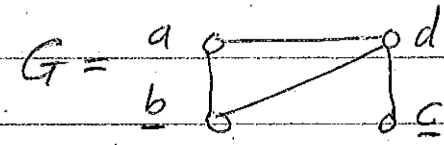
$H_4 =$

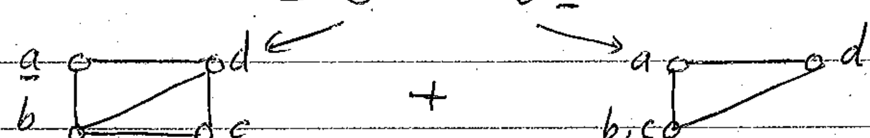


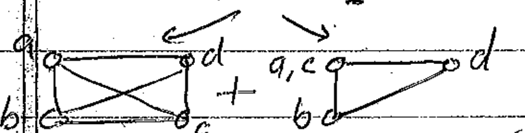
NON-PLANAR

4(a)

$G =$







$$P_G(\lambda) = P_{K_4}(\lambda) + 2P_{K_3}(\lambda)$$
$$= \lambda(\lambda-1)(\lambda-2)(\lambda-3) + 2\lambda(\lambda-1)(\lambda-2)$$
$$= \lambda(\lambda-1)(\lambda-2)[(\lambda-3)+2] = \lambda(\lambda-1)^2.(\lambda-2).$$

(b) We will prove the result by induction on $n = |V(T)|$. If $n=1$, then $T \cong K_1$. So $P_T(\lambda) = \lambda = \lambda(\lambda-1)^{1-1}$. Hence the result is true for $n=1$. Now suppose the result is true for all trees with n vertices. Let T be any tree with $n+1$ vertices and choose any leaf v_0 in T. (To get a leaf just look at the endpoints of a longest path in T.) Put $T' = T - \{v_0\}$. Then $P_T(\lambda) = (\lambda-1).P_{T'}(\lambda)$
$= (\lambda-1).\lambda.(\lambda-1)^{n-1} = \lambda.(\lambda-1)^{n+1-1}$ by the induction hypothesis. So if the result is true for n, it will be true for $n+1$. Hence the result is true for all trees by the Principle of Math Induction.

5(a) A minimum salesman walk is a closed walk of shortest possible length which includes all the vertices of G.

(b) Let H be the graph obtained by adding a new vertex v_{p+1} to G and edges from v_{p+1} to each of the vertices of G. Then H has $p+1$ vertices and for any pair of non-adjacent vertices x & y in H, we have

$$\deg_H(x) + \deg_H(y) = \{\deg_G(x)+1\} + \{\deg_G(y)+1\}$$
$$\geq (p-1)+2 = p+1.$$

So by Ore's theorem, H has a Hamilton cycle. Now if we delete the vertex v_{p+1} from C, we will get a Hamilton path P of G. Thus G has a Hamilton path.

6 (a) A simple polyhedron is a solid figure which is bounded by plane polygonal faces and which can be continnosly distorted into a solid sphere.

(b) Let $A_1, \ldots, A_r$ be the regions of a planar embedding of the polyhedron G. Since G has no triangular faces, $e(A_i) \geq 4$ for each A_i. So

$$4r = 4+4+\cdots+4 \quad (r \text{ times})$$
$$\leq e(A_1)+e(A_2)+\cdots+e(A_r)$$
$$= 2q \quad \text{because each edge is in 2 regions.}$$

$\therefore\ 2r \leq q$. Since G is a simple polyhedron, the embedding will be a connected planar graph. So by Euler's formula $r = q+2-p$. Thus

$$2(q+2-p) \leq q \implies 2q+4-2p \leq q$$
$$\implies q \leq 2p-4.$$

Thus $q(G) \leq 2p(G)-4$ and we are done.