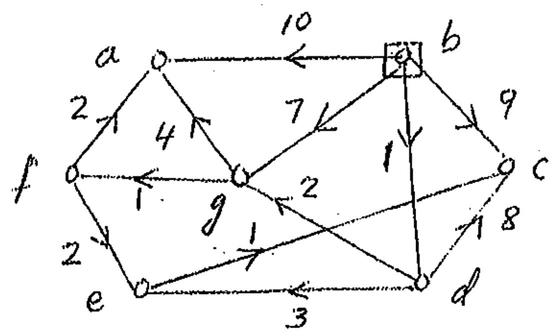
MAD 3305 - GRAPH THEORY
TEST #1 Spring 2015

FLORIDA INT'L UNIV.
TIME: 75 min.

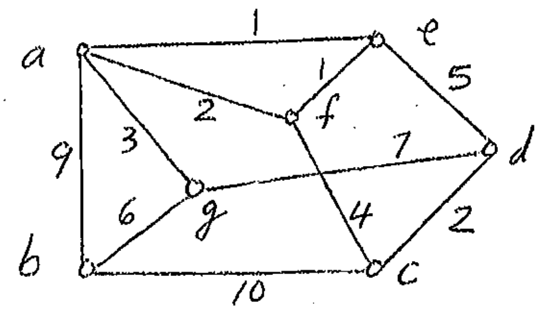
Answer *all 6* questions. ***No Calculators or Cellphones are allowed.*** *An unjustified answer will receive little or no credit. BEGIN EACH OF THE 6 QUESTIONS ON A SEPARATE PAGE.*

(15) 1. Find the *distances* from b to each of the other vertices of the graph on the right by using *Dijkstra's Algorithm.*



(20) 2(a) Find a *graph* with degree sequence $\langle 4, 3, 3, 3, 3 \rangle$ by using the *Graphical Sequence Algorithm.*

(b) For the graph on the right, find a *minimal spanning tree* by using *Prim's Algorithm* & starting at b.



(20) 3(a) Find the *tree* coded by $\langle 2, 1, 2, 4 \rangle$ via *Prufer's Tree Decoding Algorithm.*
(b) The five characters a, b, c, d, e occur with frequencies 2, 3, 4, 5, 10 respectively. Find an *optimal binary coding* for these five characters and the *weighted-path length* of your coding by using *Huffman's algorithm.*

(15) 4(a) Define what is the *adjacency matrix* of a graph G with $V(G)=\{1,2,3,\dots,p\}$.
(b) Prove that any tree with p vertices has exactly $p-1$ edges.

(15) 5(a) Define what is *the vertex-connectivity* of a graph G.
(b) Prove that if G is a disconnected graph with p vertices, then
$$|E(G)| \le (p-1).(p-2)/2.$$

(15) 6(a) Define what is a *legal flow* in a network $N = \langle G, s, t, c \rangle$.
(b) A certain tree T has 10 vertices of degree 5, 20 of degree 4, 30 of degree 3, & the rest of degree 1 or 2. What is the smallest possible value of $|V(T)|$? (*Justify your answer.*)

MAD 3305 - Graph Theory — Florida Int'l Univ.

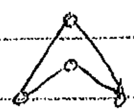
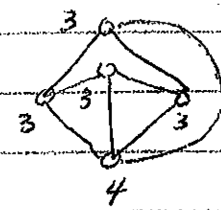
Solutions to Test #1 — Spring 2015

1.

$L(a)$	$L(b)$	$L(c)$	$L(d)$	$L(e)$	$L(f)$	$L(g)$	T	i	x_0
∞	0	∞	∞	∞	∞	∞	$\{a,b,c,d,e,f,g\}$	0	b
10	.	9	1	∞	∞	7	$\{a,c,d,e,f,g\}$	1	d
10	.	9	.	4	∞	3	$\{a,c,e,f,g\}$	2	g
7	.	9	.	4	4	.	$\{a,c,e,f\}$	3	e
7	.	5	.	.	4	.	$\{a,c,f\}$	4	f
6	.	5	.	.	.	.	$\{a,c\}$	5	c
6	.	.	.	.	.	.	$\{a\}$	6	a
.	.	.	.	.	.	.	$\emptyset$	STOP	

$d(b,\cdot) = 6 \quad 0 \quad 5 \quad 1 \quad 4 \quad 4 \quad 3$.

2(a) 4, 3, 3, 3, 3

2, 2, 2, 2 $\longrightarrow$ G_3 =  G_4 = 

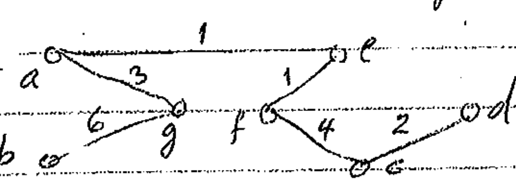
1, 1, 2

2, 1, 1 $\longrightarrow$ G_2 = 

0, 0 $\longrightarrow$ G_1 = 

(b)

a	b	c	d	e	f	g	$V(T)$	$E(T)$	i	x_0
∞	0	∞	∞	∞	∞	∞	$\{b\}$	$\emptyset$	0	b
9	.	10	∞	∞	∞	6	$\{b,g\}$	$\{bg\}$	1	g
3	.	10	7	∞	∞	.	$\{b,g,a\}$	$\{bg,ag\}$	2	a
.	.	10	7	1	2	.	$\{b,g,a,e\}$	$\{bg,ag,ae\}$	3	e
.	.	10	5	.	1	.	$\{b,g,a,e,f\}$	$\{bg,ag,ae,ef\}$	4	f
.	.	4	5	.	.	.	$\{b,g,a,e,f,c\}$	$\{bg,ag,ae,ef,fc\}$	5	c
.	.	.	2	.	.	.	$\{b,g,a,e,f,c,d\}$	$\{bg,ag,ae,ef,fc,cd\}$	6	d

T_{min} = 

3(a) $\underline{s} = \langle 2,1,2,4 \rangle \quad |\underline{s}| = p-2 \Rightarrow p = |\underline{s}|+2 = 4+2 = 6$

$d_i(1)$	$d_i(2)$	$d_i(3)$	$d_i(4)$	$d_i(5)$	$d_i(6)$	i	$\ell(i)$ — $s(i)$
2	3	1	2	1	1	1	3 — 2
2	2	0	2	1	1	2	5 — 1
1	2	0	2	0	1	3	1 — 2
0	1	0	2	0	1	4	2 — 4
0	0	0	1	0	1	5	add 4 — 6

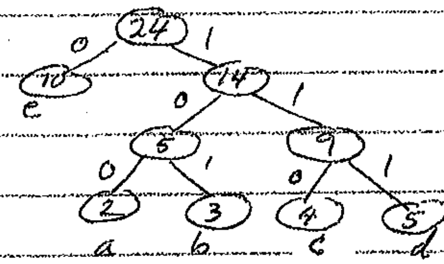
(b) 2, 3, 4, 5, 10

4, 5, 5, 10

5, 9, 10

10, 14

24.



Char.	a	b	c	d	e
Freq.	2	3	4	5	10
Codes	100	101	110	111	0
$f(c_i).\ell(c_i)$	6	9	12	15	10

Weighted path length of coding $= \sum_{i=1}^{5} f(c_i).\ell(c_i)$

$= 6+9+12+15+10 = 52.$

4(a) The matrix A is defined by $A[i,j]$ = no. of edges from i to j in G.

(b) We will prove the result by strong induction on $p = |V(T)|$.

Basis: If $p=1$, then $T \cong K_1$. So $|E(T)| = 0 = 1-1 = |V(T)|-1$. Hence the result is true for all trees with 1 vertex

Ind. Step: Suppose the result is true for all trees with $\leq p$ vertices. Let T be any tree with $p+1$ vertices. Choose any e in T and consider $T-\{e\}$. $T-\{e\}$ will be a union of two disjoint trees T_1 & T_2 because there was only one path between the endpoints of e. So $|E(T)| = |E(T_1)| + |E(T_2)| + |\{e\}|$

$= |V(T_1)|-1 + |V(T_2)|-1+1 = |V(T)|-1.$

So if the result is true for all trees with $\leq p$ vertices, it will be true for all trees with $p+1$ vertices. By the strong Principle of ~~Math Induction, the result now follows for all trees.~~

5(a) The vertex-connectivity of G is defined by $k_v(G)$ = smallest no. of vertices whose removal will disconnect G or will reduce it to K_1.

(b) Since G is a disconnected graph we can split G into two non-empty parts G_1 & G_2 such that there are no edges from G_1 to G_2. Let $k = |V(G_1)|$. Then $|V(G_2)| = p-k$. Since G_1 & G_2 has k & $p-k$ vertices, $|E(G_1)| \le k(k-1)/2$ & $|E(G_2)| \le (p-k)(p-k-1)/2$. So $|E(G)| = |E(G_1)| + |E(G_2)| \le [k(k-1) + (p-k)(p-k-1)]/2$. Thus

$$(p-1)(p-2)/2 - |E(G)| \ge [(p-1)(p-2) - k(k-1) - (p-k)(p-k-1)]/2$$
$$= (p^2 - 3p + 2 - k^2 + k - p^2 + pk + p + pk - k^2 - k)/2$$
$$= (2pk - 2p - 2k^2 + 2)/2 = pk - p - k^2 + 1$$
$$= (p-k-1)(k-1) \ge 0 \quad \text{because } k \ge 1 \text{ \& } p-k \ge 1.$$

Hence $E(G) \le (p-1)(p-2)/2$.

6(a) A legal flow in N is any function $f: E(G) \to [0, \infty)$ such that (i) $f(e) \le c(e)$ for each $e \in E(G)$, and (ii) $\sum_{e \in In(v)} f(e) = \sum_{e \in Out(v)} f(e)$ for each $v \in V(G) - \{s, t\}$.

(b) Let k = no. of vertices of deg. 2 in T and l = no. of leaves in T. Then $|V(T)| = 10 + 20 + 30 + k + l = 60 + k + l$

Also sum of degrees in $T = 2|E(T)| = 2\{|V(T)| - 1\} = 118 + 2k + 2l$

But sum of degrees in $T = 10(5) + 20(4) + 30(3) + 2(k) + 1(l)$
$= 50 + 80 + 90 + 2k + l = 220 + 2k + l$

$\therefore\ 118 + 2k + 2l = 220 + 2k + l$

$\therefore\ l = 220 - 118 = 102$.

Now by merging out all the vertices of deg 2 in any T with 10 vertices of deg. 5, 20 of deg. 4, & 30 of degree 3, we will end up with a tree with no vertices of degree 2.

So minimum possible value of $|V(T)| = 60 + 0 + 102 = 162$.

END